Light-Matter Interaction

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Foreword: The Quantum Conversation between Light and Matter

The interaction between light and matter is the fundamental dialogue of the physical universe. Nearly every phenomenon we observe—from the biological process of vision to the high-speed data transmission in laser and fiber-optic networks—is a manifestation of **charged particles** accelerating in **time-dependent electric fields**. At its core, this interaction describes how an accelerating charge generates light and, conversely, how electric fields exert forces on matter.

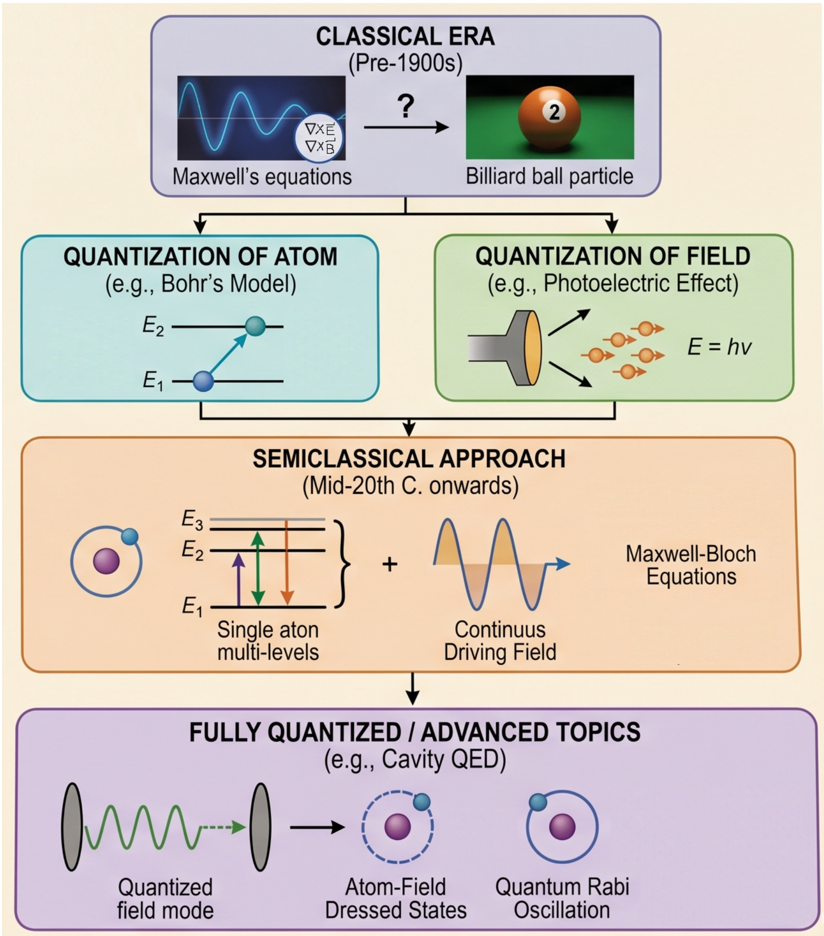

Figure 1: Schematics of quantum light-matter interaction.

The Two Primary Actors

To understand this dialogue, we must first define our participants. In this note, matter is described by an electron in an N-level system, most often simplified to a two-level system (TLS) with energies E_c and E_v. The Hamiltonian for matter is expressed as:

$$\mathcal{H}_0 = \sum_n E_n |E_n\rangle\langle E_n|$$

The electromagnetic field, our second actor, is treated as a collection of harmonic oscillators. Even in a vacuum, each mode possesses a zero-point energy, leading to a field Hamiltonian:

$$\mathcal{H}_F = \sum_m \hbar\omega_m (a_m^\dagger a_m + \frac{1}{2})$$

The Roadmap of This Note

We explore this interaction through three increasing levels of theoretical sophistication:

- **Semi-Classical Approach**: We treat matter as quantum mechanical, while the electric field is a classical number. This framework allows us to derive *Rabi Oscillations*the coherent sloshing of probability between statesand *Fermi's Golden Rule*, which describes incoherent transitions and the density of states.
- **Classical Matter and Quantum Fields**: We reverse the roles, treating macroscopic currents as classical sources that drive the quantum field. This transition reveals how simple harmonic currents can generate *Coherent States*, the most "classical-like" states of light.
- **Fully Quantum Approach**: Finally, we quantize both actors simultaneously. This leads us to the *Jaynes-Cummings Model*, the crown jewel of Cavity Quantum Electrodynamics (CQED).

Motivation: Why Quantize Matter and Light?

Before diving into the mathematics of interaction, it is crucial to ask: why do we need a quantum description in the first place? Why can't we simply rely on Maxwell's equations and Newton's laws?

The Necessity of Quantum Matter: In a microscopic world, electron motion is confined, leading to discrete energy levels and wave functions. Classical quantities, such as charge and current densities, should be replaced by their quantum counterparts. We must restrict the electrons to discrete, stationary energy levels, such as the $|E_c\rangle$ and $|E_v\rangle$ states of our two-level system.

The Necessity of Quantum Light: In classical electrodynamics, an electromagnetic wave is described by a continuous amplitude and a well-defined phase ($e^{-i\omega t}$). However, quantum optics reveals that this classical description is merely a special case—specifically, it corresponds to a macroscopic *coherent state* of photons. The true quantum reality of light is vastly richer. A photonic system occupies a Hilbert space with an infinite number of possibilities beyond coherent states, such as discrete number states ($|n\rangle$), squeezed states, and entangled superpositions. Recognizing the fully quantized nature of light is essential, as these strictly quantum states unlock new regimes of light-matter interaction that are fundamentally impossible to describe with classical fields.

Time-dependent charges can be described by a charge density $\rho(\mathbf{r}, t)$. Dipoles and currents are more commonly used to describe light-matter interactions. Polarization $\mathbf{P}$ (dipole) and currents density $\mathbf{J}$ have the relations

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0, \tag{0.1}$$

$$\mathbf{J} = \frac{\partial \mathbf{P}}{\partial t}. \tag{0.2}$$

1 Hamiltonian

1.1 Interaction Hamiltonian

According to classical mechanics, a charged particle has the Hamiltonian (SI units)

$$\mathcal{H} = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} + q\Phi(\mathbf{r}, t), \tag{1.1}$$

where q is the charge of the particle, not the position. $\Phi(\mathbf{r}, t)$ is the electric potential. In the case of an electron, $q = -e$, we have

$$\mathcal{H} = \frac{(\mathbf{p} + e\mathbf{A})^2}{2m} - e\Phi(\mathbf{r}, t). \tag{1.2}$$

We can decompose it into $\mathcal{H}_0$ and $\mathcal{H}_I$,

$$\mathcal{H}_0 = \frac{p^2}{2m}, \tag{1.3}$$

$$\mathcal{H}_I = \frac{e\left(\mathbf{p} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{p}\right)}{2m} + \frac{e^2 A^2}{2m} - e\Phi. \tag{1.4}$$

Typically, the term $\frac{e^2 A^2}{2m}$ is dropped since the momentum of the field $e\mathbf{A}$ is usually smaller than the electron's momentum $\mathbf{p}$.[1] Since the momentum $\mathbf{p}$ is a differential operator, $\mathbf{p}\cdot\mathbf{A}$ is not equal to $\mathbf{A}\cdot\mathbf{p}$. The vector potential $\mathbf{A}$ and Coulomb's potential Φ are not unique. Maxwell's equations are invariant under the gauge transformations

$$\mathbf{A}' = \mathbf{A} + \nabla\lambda(\mathbf{r}, t), \tag{1.5}$$

$$\Phi' = \Phi - \frac{\partial \lambda(\mathbf{r}, t)}{\partial t}. \tag{1.6}$$

The fields are given by

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{1.7}$$

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{1.8}$$

[1] Well, this is a sloppy argument. In electromagnetism, the source of the vector potential $\mathbf{A}$ is current, so $\mathbf{A}$ is proportional to $\frac{v}{c}$, where v is the electron speed and c is the light speed. The term $\frac{e^2 A^2}{2m}$ is proportional to $\frac{v^2}{c^2}$, which is typically small.

To simplify the Hamiltonian, the Coulomb gauge ($\nabla \cdot \mathbf{A}$) with the condition $\Phi = 0$.[2] is used most of the time. In the the Coulomb gauge, using $\mathbf{p} = -i\hbar\nabla$, we have

$$\mathbf{p} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{p}$$

The interaction Hamiltonian becomes

$$\mathcal{H}_I = \frac{e(\mathbf{A} \cdot \mathbf{p})}{m} \tag{1.9}$$

$$= -\int dv \mathbf{A} \cdot \mathbf{J} \tag{1.10}$$

where we use $\int dv \mathbf{J} = \frac{-e\mathbf{p}}{m}$.

The interaction Hamiltonian is now related to current and vector potential. It is possible to replace current and vector potential by dipoles and electric fields. We use the Göppert-Mayer gauge,

$$\lambda = -(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{A}(\mathbf{r}_0). \tag{1.11}$$

Using this gauge and Eq. (1.6), we have

$$\mathbf{A}' = \mathbf{A}(\mathbf{r}) - \mathbf{A}(\mathbf{r}_0), \tag{1.12}$$

$$-e\Phi' = e(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{E}(\mathbf{r}_0) \equiv -\mathbf{d} \cdot \mathbf{E}, \tag{1.13}$$

where $\mathbf{d} = -e(\mathbf{r} - \mathbf{r}_0)$ is the dipole operator since $\mathbf{r}$ is the position operator. The so-called dipole approximation is when $\mathbf{A}(\mathbf{r})$ is almost a constant, i.e., $\mathbf{A}(\mathbf{r}) \simeq \mathbf{A}(\mathbf{r}_0)$. In this approximation, the new vector potential $\mathbf{A}'(\mathbf{r}_0)$ vanishes. This approximation is valid if the field changes gradually over the range of the charge distributions. For example, the charge distribution of an atom is about 0.1 nm, and the electric field of visible light is almost constant over the atom since the wavelengths range from 400 to 700 nm. The interaction Hamiltonian becomes

$$\mathcal{H}_I = -\mathbf{E} \cdot \mathbf{d} \tag{1.14}$$

Although we did not define the field operator $\mathbf{A}$, it can be obtained by the relation of the electric field operator and the vector potential operator

$$\boldsymbol{\mathcal{E}} = -\frac{\partial}{\partial t}\boldsymbol{\mathcal{A}} \tag{1.15}$$

$$= i\omega\boldsymbol{\mathcal{A}} \tag{1.16}$$

$$\mathbf{A} = \left(\frac{\mathcal{E}a - \mathcal{E}^* a^\dagger}{2i\omega}\right). \tag{1.17}$$

1.2 Total Hamiltonian

The total Hamiltonian of the light-matter system is

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I + \mathcal{H}_F. \tag{1.18}$$

[2] In the region without charges $\nabla \cdot \mathbf{E} = 0$, we can define $\mathbf{E} = -\nabla\Phi$. Using the gauge transformation $\lambda = \int \Phi dt$, we can eliminate Φ and make $\nabla \cdot \mathbf{A} = 0$.

where

$$\mathcal{H}_F = \sum_m \int dv \left(\frac{\epsilon(\mathbf{r})E_m^2(\mathbf{r})}{2} + \frac{B_m^2(\mathbf{r})}{2\mu(\mathbf{r})} \right) \tag{1.19}$$

$$= \sum_m \hbar\omega_m \left(a_m^\dagger a_m + \frac{1}{2} \right). \tag{1.20}$$

The Hamiltonian of matter $\mathcal{H}_0$ is not necessarily in the form of a free particle. In general, $\mathcal{H}_0$ describes a N-level system,

$$\mathcal{H}_0 = \sum_n E_n |E_n\rangle\langle E_n|. \tag{1.21}$$

The simplest case is a two-level system (TLS)

$$\mathcal{H}_{TLS} = \begin{pmatrix} E_c & 0 \\ 0 & E_v \end{pmatrix}. \tag{1.22}$$

The interaction Hamiltonian for a two-level system is

$$\mathcal{H}_I = \begin{pmatrix} \langle E_c| - \mathbf{E}\cdot\mathbf{d}|E_c\rangle & \langle E_c| - \mathbf{E}\cdot\mathbf{d}|E_v\rangle \\ \langle E_v| - \mathbf{E}\cdot\mathbf{d}|E_c\rangle & \langle E_v| - \mathbf{E}\cdot\mathbf{d}|E_v\rangle \end{pmatrix} \tag{1.23}$$

$$= -\mathbf{E}\cdot \begin{pmatrix} \mathbf{d}_{cc} & \mathbf{d}_{cv} \\ \mathbf{d}_{vc} & \mathbf{d}_{vv} \end{pmatrix}, \tag{1.24}$$

where the dipole matrix element is $\mathbf{d}_{nn'} = \langle E_n|\mathbf{d}|E_{n'}\rangle$. In many cases, the diagonal elements of dipole matrices vanish since the charge densities of the eigenfunctions are typically symmetric.

2 Classical Fields and Quantum Matter

We consider that the matter is described by an N-level system and treat the electric field $\mathbf{E}(\mathbf{r},t)$ as a number. The Hamiltonian is

$$\mathcal{H} = \sum_n E_n|E_n\rangle\langle E_n| - \mathbf{E}\cdot\mathbf{d}. \tag{2.1}$$

In the case of a TLS system, the Hamiltonian is

$$\mathcal{H} = \begin{pmatrix} E_c & 0 \\ 0 & E_v \end{pmatrix} - \mathbf{E}\cdot \begin{pmatrix} 0 & \mathbf{d}_{cv} \\ \mathbf{d}_{vc} & 0 \end{pmatrix}, \tag{2.2}$$

where we assume the diagonal elements of the dipole matrix are zeros. To solve the dynamics, we start with the interaction picture, where the state is

$$|\psi\rangle = C_c(t)e^{-i\omega_c t}|E_c\rangle + C_v(t)e^{-i\omega_v t}|E_v\rangle. \tag{2.3}$$

It is clear that without an external field $\mathbf{E}$, the coefficients $C_c(t)$ and $C_v(t)$ are constant in time. Plugging Eq. (2.3) in the Schrödinger equation, we obtain

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} C_c \\ C_v \end{pmatrix} = -\mathbf{E}\cdot \begin{pmatrix} 0 & \mathbf{d}_{cv}e^{i(\omega_c-\omega_v)t} \\ \mathbf{d}_{vc}e^{i(\omega_v-\omega_c)t} & 0 \end{pmatrix} \begin{pmatrix} C_c \\ C_v \end{pmatrix}. \tag{2.4}$$

The dipole matrix elements in the interaction picture oscillate rapidly in time. The electric field $\mathbf{E} = \boldsymbol{\mathcal{E}}_\omega e^{-i\omega t} + \boldsymbol{\mathcal{E}}_\omega^* e^{i\omega t}$ needs to have a frequency $\omega \simeq (\omega_c - \omega_v)$ in order to create transition. We write

$$\omega = \omega_{cv} + \Delta, \tag{2.5}$$

where $\omega_{cv} = \omega_c - \omega_v$ and Δ is the detuning.

2.1 Rabi Model

Let the external field $\mathbf{E} = \mathbf{E}_0 \cos\omega t = \mathbf{E}_0\left(\frac{e^{-i\omega t}+e^{i\omega t}}{2}\right)$. The equation of the coefficients is

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} C_c \\ C_v \end{pmatrix} = \begin{pmatrix} 0 & \frac{V_0}{2}\left[e^{-i\Delta t} + e^{i(2\omega_{cv}+\Delta)t}\right] \\ \frac{V_0^*}{2}\left[e^{i\Delta t} + e^{-i(2\omega_{cv}+\Delta)t}\right] & 0 \end{pmatrix}\begin{pmatrix} C_c \\ C_v \end{pmatrix}. \tag{2.6}$$

where

$$V_0 = -\mathbf{E}_0 \cdot \mathbf{d}_{cv}. \tag{2.7}$$

The equation needs to be solved numerically. The rotating-wave approximation (RWA), which drops high-frequency terms, is often used. Under the RWA, the equation reads

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} C_c \\ C_v \end{pmatrix} = \begin{pmatrix} 0 & \frac{V_0}{2}e^{-i\Delta t} \\ \frac{V_0^*}{2}e^{i\Delta t} & 0 \end{pmatrix}\begin{pmatrix} C_c \\ C_v \end{pmatrix}. \tag{2.8}$$

Eliminating the variable C_v, we obtain the second-order differential equation

$$\ddot{C}_c + i\Delta\dot{C}_c + \frac{|V_0|^2}{4\hbar^2}C_c = 0. \tag{2.9}$$

The general solution is

$$C_c(t) = A_+e^{i\lambda_+ t} + A_-e^{i\lambda_- t} \tag{2.10}$$

with

$$\lambda_\pm = \Delta \pm \sqrt{\Delta^2 + \frac{|V_0|^2}{\hbar^2}} \equiv \Delta \pm \Omega_R. \tag{2.11}$$

The Rabi frequency $\Omega_R = \sqrt{\Delta^2 + \frac{|V_0|^2}{\hbar^2}}$. If initially $C_v(0) = 1$ and $C_c(0) = 1$, the solution is

$$C_c = e^{i\frac{\Delta t}{2}}\frac{iV_0}{\hbar\Omega_R}\sin\frac{\Omega_R t}{2}, \tag{2.12}$$

$$C_v = e^{i\frac{\Delta t}{2}}\left[\cos\frac{\Omega_R t}{2} - i\frac{\Delta}{\Omega_R}\sin\frac{\Omega_R t}{2}\right]. \tag{2.13}$$

It can be checked that $|C_c|^2 + |C_v|^2 = 1$. The population of the excited state is

$$P_c(t) = |C_c(t)|^2 = \frac{|V_0|^2 \sin^2 \frac{\Omega_R t}{2}}{\hbar^2 \Omega_R^2} \tag{2.14}$$

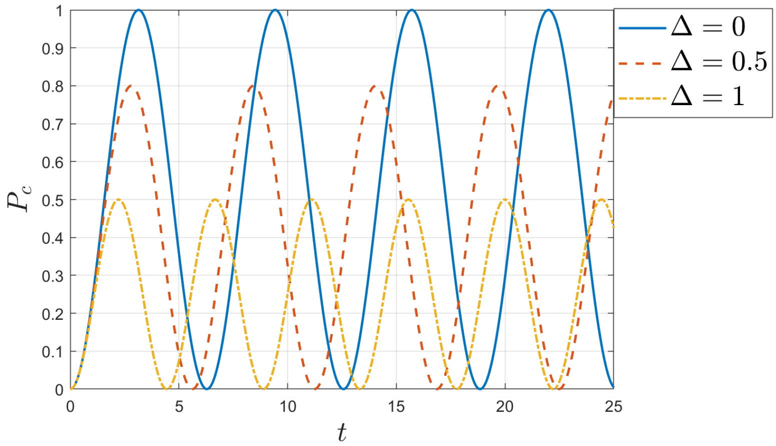

Figure 2: Population of the excited state as a function of time with $\frac{V_0}{\hbar} = 1$.

2.2 Fermi's Golden Rule

If the external field is small, we can obtain from Eq. (2.12) [3]

$$P_c(t) = |C_c|^2 = \frac{|V_0|^2 \sin^2 \frac{\Delta t}{2}}{\hbar^2 \Delta^2}. \tag{2.15}$$

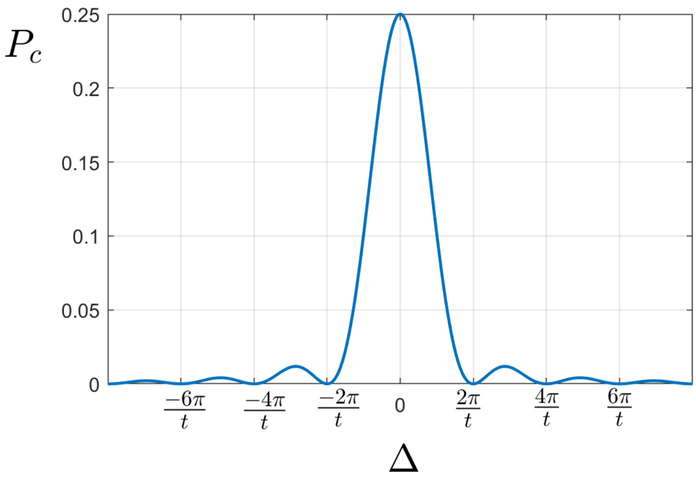

Figure 3: The transition probability $P_c(t)$ at a momentum t. When t is large, the function is approximately a delta function.

[3] the formal method to obtain this result is the the perturbation method (for example, see Chapter 5 of Ref. [1])

When t is large, the fraction is approximately a delta function

$$\frac{\sin^2\frac{\Delta t}{2}}{\Delta^2} \simeq \frac{\pi t}{2}\delta(\Delta). \tag{2.16}$$

The transition rate $W_{v\to c}$ is

$$W_{v\to c} = \frac{P_c(t)}{t} = \frac{\pi}{2}\frac{|V_0|^2}{\hbar^2}\delta(\omega-\omega_{cv}) \tag{2.17}$$

$$= \frac{\pi}{2}\frac{|\mathbf{E}_0\cdot\mathbf{d}_{cv}|^2}{\hbar^2}\delta(\omega-\omega_{cv}) \tag{2.18}$$

$$= \frac{\pi}{2}\frac{|\langle c|\mathbf{H}_I|v\rangle|^2}{\hbar^2}\delta(\omega-\omega_{cv}), \tag{2.19}$$

which is the famous Fermi's Golden Rule. The unit of $\delta(\omega-\omega_{cv})$ is one over frequency. The delta function $\delta(\omega-\omega_{cv})$ is interpreted as the density of states. Since we consider only a two-level system, there is only one final state for $\omega_{cv} - d\omega/2 < \omega < \omega_{cv} + d\omega/2$. If instead, we consider there are many states between $\omega_{cv} - d\omega/2$ and $\omega_{cv} + d\omega/2$, we will use the the density of states $\rho(\omega)$, defined by

$$\rho(\omega) = \frac{dN}{d\omega}, \tag{2.20}$$

where N is the number of states between $\omega_{cv} - d\omega/2$ and $\omega_{cv} + d\omega/2$. In this case, Fermi's Golden Rule becomes

$$W = \frac{\pi}{2}\frac{|\langle c|\mathbf{H}_I|v\rangle|^2}{\hbar^2}\rho(\omega), \tag{2.21}$$

or, in terms of energies,

$$W = \frac{\pi}{2}\frac{|\langle c|\mathbf{H}_I|v\rangle|^2}{\hbar}\rho(E), \tag{2.22}$$

where $\rho(E)dE$ is the number of states for E between $E_{cv} - dE/2$ and $E_{cv} + dE/2$.

Using the Fermi Golden rule, one can derive the famous rate of spontaneous emission in a vacuum,

$$W_{\text{sp}} = \frac{\omega^3|d_{cv}|^2}{3\pi\epsilon_0\hbar c^3}. \tag{2.23}$$

Note 1: Fermi's Golden Rule

- Fermi's golden rules are valid in the perturbation regime ($|\langle c|\mathbf{H}_I|v\rangle|$ is small compared to E_{cv})
- Fermi's golden rules describe the incoherent excitation. The excitation events are independent, and the final state is almost empty. These conditions are not true for the Rabi oscillation (coherent excitation).
- The rate is proportional to the square of the transition dipole element $|\langle c|\mathbf{d}|v\rangle|^2$

- The rate is proportional to the density of the final states.

2.3 Density Matrix Approach

Consider a classical light interacting with an ensemble of the same two-level systems. We need to use the density matrix

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \tag{2.24}$$

and use the quantum Liouville's equation to obtain

$$\frac{d\rho_{11}}{dt} = i\Omega_R(\rho_{12} - \rho_{21}), \tag{2.25}$$

$$\frac{d\rho_{22}}{dt} = -i\Omega_R(\rho_{12} - \rho_{21}), \tag{2.26}$$

$$\frac{d\rho_{12}}{dt} = i\Omega_R(\rho_{11} - \rho_{22}), \tag{2.27}$$

$$\frac{d\rho_{21}}{dt} = -i\Omega_R(\rho_{11} - \rho_{22}), \tag{2.28}$$

where ρ_{11} and ρ_{22} describe probabilities, and ρ_{12} and ρ_{21} describe coherence. Further simplifications give

$$\frac{d^2\rho_{11}}{dt^2} = -2\Omega_R^2(\rho_{11} - \rho_{22}) \tag{2.29}$$

$$= -2\Omega_R^2(2\rho_{11} - 1). \tag{2.30}$$

2.4 Optical Bloch Equations

In the previous sections, we treated an idealized, isolated two-level system. Under the pure-state Schrödinger equation, a constant driving field results in eternal Rabi oscillations (as seen in Eq. (2.15)). However, real atoms are open systems constantly interacting with the vacuum and their surrounding environment, leading to dissipation and decoherence. To account for this statistical mixture of states, we must transition from the pure-state Liouville equation to the **Optical Bloch Equations (OBEs)**.

We introduce two phenomenological relaxation parameters to capture environmental coupling:

- **Longitudinal Relaxation Rate (Γ):** The rate at which the excited state population (ρ_{22}) decays to the ground state (ρ_{11}), primarily due to spontaneous emission. The associated lifetime is $T_1 = 1/\Gamma$.
- **Transverse Relaxation Rate (γ):** The rate at which the quantum phase coherence (ρ_{12} and ρ_{21}) decays. This includes both the fundamental limit imposed by spontaneous emission ($\Gamma/2$) and pure dephasing caused by atomic

collisions or laser phase fluctuations. The associated coherence time is $T_2 = 1/\gamma$.

Adding these relaxation terms and the laser detuning Δ to the pure-state dynamics, we obtain the full Optical Bloch Equations in the rotating frame:

$$\frac{d\rho_{11}}{dt} = \Gamma\rho_{22} + i\Omega_R(\rho_{12} - \rho_{21})$$

$$\frac{d\rho_{22}}{dt} = -\Gamma\rho_{22} - i\Omega_R(\rho_{12} - \rho_{21})$$

$$\frac{d\rho_{12}}{dt} = -(\gamma + i\Delta)\rho_{12} + i\Omega_R(\rho_{11} - \rho_{22})$$

$$\frac{d\rho_{21}}{dt} = -(\gamma - i\Delta)\rho_{21} - i\Omega_R(\rho_{11} - \rho_{22})$$

Unlike the pure Rabi model, the OBEs allow the system to reach a *steady state* (where $d\rho/dt = 0$). Solving for these steady-state values is the foundational step for deriving the natural lineshape of atomic transitions and understanding continuous-wave (CW) laser interactions.

2.5 Closing the Loop: The Maxwell-Bloch Equations

The Optical Bloch Equations describe how a classical field drives a quantum atom. However, a complete semiclassical theory must close the loop: the atoms must also react back on the field.

When a laser propagates through an atomic medium, it interacts not with one atom, but with a macroscopic ensemble. The off-diagonal coherence terms (ρ_{12} and ρ_{21}) correspond to an oscillating microscopic quantum dipole. For an ensemble with an atomic density N, these microscopic dipoles sum to create a macroscopic **polarization density**, $P(t)$:

$$P(t) = N\mathrm{Tr}(\rho d) = N(d_{12}\rho_{21} + d_{21}\rho_{12})$$

This quantum-mechanically derived polarization acts as the driving source term in the classical Maxwell wave equation:

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P}{\partial t^2}$$

Coupling the OBEs with the Maxwell wave equation yields the **Maxwell-Bloch Equations**. This is the ultimate mathematical framework for the semiclassical regime, describing precisely how light pulses dynamically reshape, amplify, or dissipate as they propagate through a resonant quantum medium.

2.6 Advanced Semiclassical Phenomena (Presentation Topics)

The Maxwell-Bloch framework enables modeling a vast array of modern quantum-optical phenomena that cannot be explained by single-atom physics alone. The following topics represent foundational breakthroughs in quantum optics and serve as excellent subjects for further exploration and presentation.

a. Nonlinear Quantum Optics and Self-Induced Transparency (SIT)
At low light intensities, an atomic medium absorbs light linearly. But what happens when the field becomes incredibly strong? The light deeply saturates the atomic transition ($\rho_{11} \approx \rho_{22}$), fundamentally altering the medium's refractive index. A spectacular time-domain result of this nonlinearity is *Self-Induced Transparency*. If a highly intense, ultrashort pulse (specifically, a 2π pulse) enters an opaque resonant medium, it continuously drives the atoms into the excited state and perfectly stimulates them back down. The pulse travels through the material entirely unattenuated as an ***optical soliton***, briefly borrowing energy from the atoms and returning it without a single photon lost.

b. Gain, Lasing, and Population Inversion
Under normal thermal equilibrium, atoms prefer the ground state ($\rho_{11} > \rho_{22}$), resulting in net light absorption. However, if external pumping mechanisms are introduced to the OBEs, we can engineer a ***population inversion*** where $\rho_{22} > \rho_{11}$. In this unnatural regime, stimulated emission outpaces absorption. The macroscopic polarization $P(t)$ from the Maxwell-Bloch equations then constructively amplifies the classical field $E(t)$. This dynamic is the theoretical engine behind laser gain and the foundation of modern optics.

c. Superradiance (Collective Emission)
If you excite a sparse gas of N atoms, they will independently decay over a time T_1, emitting a faint glow with an intensity proportional to N. But in 1954, Robert Dicke asked: what if those N atoms are packed into a space smaller than the wavelength of the light they emit? The Maxwell-Bloch coupling forces their dipoles to phase-lock and synchronize. They act as a single giant "super-atom," emitting a brief, blindingly intense, and highly directional flash of light. The intensity scales as N^2, and the decay time plummets to $1/N$—a profound demonstration of collective quantum behavior.

d. Optical Bistability
Can light act like a computer's memory? When a nonlinear resonant medium is placed inside an optical cavity (between mirrors), the light continuously feeds back into the atoms. Under the right conditions, a single, constant input intensity can result in two completely distinct, stable output states—say, 10% transmission or 90% transmission. This creates a hysteresis loop, allowing the atomic system to act as an all-optical switch. It is a foundational concept for researchers attempting to build light-based quantum and classical computers.

e. Coherent Transients: Reversing Time with Photon Echoes
The OBEs are uniquely equipped to model transient "memory" effects. If a laser driving an ensemble is suddenly snapped off, the macroscopic polarization doesn't vanish instantly; it rings like a struck bell, radiating a decaying field called *Free Induction Decay* (FID). Because real atoms exist in different environments, they have slightly different resonance frequencies. Their dipoles rapidly dephase, washing out the macroscopic signal. Amazingly, applying a second, precisely timed laser pulse can physically reverse this quantum dephasing. The dipoles rewind, perfectly re-aligning to emit a spontaneous, intense burst of light known as a *Photon Echo*.

f. Three-Level Quantum Interference and Control
By expanding the Optical Bloch Equations from two levels to three (a Λ-type system), entirely new pathways of quantum interference emerge:

- **Coherent Population Trapping (CPT)**: By illuminating an atom with two carefully tuned lasers, the electron can be trapped in a "dark state"—a fragile quantum superposition of the two lower energy levels that fundamentally refuses to absorb light.
- **Electromagnetically Induced Transparency (EIT)**: The macroscopic manifestation of CPT. A medium that is normally pitch-black to a "probe" laser is suddenly rendered perfectly transparent by turning on a second "coupling" laser. EIT also drastically steepens the refractive index, leading to *slow light*, where a photon pulse's group velocity can be throttled down to the speed of a moving bicycle.
- **Stimulated Raman Adiabatic Passage (STIRAP)**: The ultimate quantum sleight-of-hand. By pulsing lasers in a highly counter-intuitive, mathematically rigorous sequence, physicists can smoothly transfer an electron from one state to another with near 100% efficiency, completely bypassing the lossy excited state along the way.

3 Classical Matter and Quantum Fields

Currents and charges are treated as classical numbers. Time-dependent charges and currents are not independent variables. They are related by the continuity equation. This assumption is adequate when the currents are due to many electrons and quantum fluctuations are ignored. The typical problem is how a current source $\mathbf{I}(\mathbf{r}, t)$ interacts with photons. The current is a control and macroscopic parameter that can be treated classically as a number. Thus, currents are given functions, and the problem is to solve the field Hamiltonian.

$$\begin{aligned}
\mathcal{H} &= \mathcal{H}_F + \mathcal{H}_I && (3.1)\\
&= \sum_m \hbar\omega_m a_m^\dagger a_m - \sum_m \mathbf{E}_m \cdot \mathbf{d} && (3.2)\\
&= \sum_m \hbar\omega_m a_m^\dagger a_m - \sum_m \left(\frac{\mathcal{E}_m a + \mathcal{E}_m^* a^\dagger}{2}\right) \cdot \mathbf{d}, && (3.3)
\end{aligned}$$

The above interaction Hamiltonian has the dipole instead of a current. Dynamically, dipoles and currents are related. Let the current be $\mathbf{I}(\mathbf{r},t) = \mathbf{I}(\mathbf{r})_0 e^{-i\omega t}$. The current is related to the current density $\mathbf{J}$ by

$$\mathbf{J}(\mathbf{r},t) = \frac{\mathbf{I}(\mathbf{r},t)}{da_\perp}. \tag{3.4}$$

From this relation, we can find the current density $\mathbf{J}(\mathbf{r},t) = \mathbf{J}_0(\mathbf{r})e^{-i\omega t}$. Now we can use the interaction Hamiltonian in terms of $\mathbf{J}$ and $\mathbf{A}$. Considering a single mode and $\omega_m = \omega$, the Hamiltonian becomes

$$\mathcal{H} = \hbar\omega a^\dagger a - \int dv \mathbf{A}\cdot\mathbf{J}. \tag{3.5}$$

In this regime, we treat macroscopic currents and charges as classical numbers, while the electromagnetic field remains quantized. While it may seem counterintuitive to use a classical source in a quantum note, this section provides the mathematical foundation for the **Displacement Operator**—the fundamental tool for manipulating quantum states in phase space.

3.1 Generation of Coherent States

We are going to show a coherent state $|\alpha\rangle$ can be generated by a harmonic oscillating current density

$$\mathbf{J} = \frac{\mathbf{J}_0(\mathbf{r})e^{-i\omega t} + \mathbf{J}_0^*(\mathbf{r})e^{i\omega t}}{2} \tag{3.6}$$

This current density oscillating with the frequency ω can excite photons of the same frequency. The total Hamiltonian is

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I, \tag{3.7}$$

with the photon Hamiltonian $\mathcal{H}_0 = \hbar\omega a^\dagger a$ and $\mathcal{H}_I = -\int dv \mathbf{A}\cdot\mathbf{J}$. Using Eq. (1.17) and the RWA, the interaction Hamiltonian becomes

$$\mathcal{H}_I = \left(V_0 a + V_0^* a^\dagger\right), \tag{3.8}$$

where

$$V_0(t) = \frac{e^{i\omega t}\int dv \boldsymbol{\mathcal{E}}_\omega^*(\mathbf{r})\cdot\mathbf{J}_0(\mathbf{r})}{4i\omega}. \tag{3.9}$$

Now, the term $V_0(t)$ is time-dependent. We can use the interaction picture to remove the time dependence. In the interaction picture[4], the interaction Hamiltonian becomes[5]

$$\tilde{\mathcal{H}}_I = \left(V_I a + V_I^* a^\dagger\right), \tag{3.10}$$

[4]Rotating with the $\mathcal{H}_0$.

[5]To avoid confusion, we use $\tilde{\mathcal{H}}_I$ to denote the interaction Hamiltonian in the interaction picture.

where the interaction potential becomes time-independent and reads

$$V_I = \frac{\int dv \boldsymbol{\mathcal{E}}^*_\omega(\mathbf{r}) \cdot \mathbf{J}_0(\mathbf{r})}{4i\omega}. \tag{3.11}$$

The evolution of a state is given by

$$|\psi(t)\rangle_I = \hat{\mathcal{T}}[e^{-i\int \frac{\tilde{\mathcal{H}}_I(t)}{\hbar}dt}]|\psi(0)\rangle_I \tag{3.12}$$

where $\hat{\mathcal{T}}[]$ denotes the time-ordering[6]. In this case, the interaction Hamiltonian in the interaction picture is time-independent,

$$|\psi(t)\rangle_I = e^{-i\frac{\tilde{\mathcal{H}}_I(t)}{\hbar}t}|\psi(0)\rangle_I \tag{3.13}$$

$$= e^{\alpha^* a - \alpha a^\dagger}|\psi(0)\rangle_I, \tag{3.14}$$

where

$$\alpha = i\frac{V_I^*}{\hbar}t. \tag{3.15}$$

Equation (3.14) is indeed the displacement operator. If the initial state is the ground state $|0\rangle$, the final state is a coherent state,

$$|\psi(t)\rangle_I = e^{\alpha^* a - \alpha a^\dagger}|0\rangle \tag{3.16}$$

$$= |\alpha\rangle. \tag{3.17}$$

One interesting observation is that $|\alpha| \sim t$ and the photon number $n \sim t^2$ grow quadratically.

3.2 The Displacement Operator: Moving the Vacuum

As derived in Eq. (3.14), a harmonic oscillating current density $\mathbf{J}(r,t)$ acting on the vacuum state $|0\rangle$ generates a coherent state $|\alpha\rangle$:

$$|\psi(t)\rangle_I = e^{\alpha a^\dagger - \alpha^* a}|0\rangle = \hat{D}(\alpha)|0\rangle = |\alpha\rangle$$

where the displacement parameter α is directly proportional to the strength and duration of the classical drive. In modern quantum technologies, $\hat{D}(\alpha)$ is not just a theoretical curiosity; it is the operational command for a "classical pulse." Whether we are sending a microwave pulse into a superconducting resonator or a laser pulse onto a trapped ion, we are applying a displacement operator to the quantum field.

[6]Time-ordering is necessary if H_I is time-dependent and $[H_I(t_1), H_I(t_2)] \neq 0$

3.3 Application: State Tomography and Probing

In current research, we often use the classical-quantum interface to perform **Wigner Function Tomography**. By mixing a "classical" coherent state $|\alpha\rangle$ with an unknown quantum state (like a Fock state $|n\rangle$) on a beam splitter, we can shift the quantum state in phase space. Measuring the resulting displacement allows us to map the entire distribution of the quantum state.

4 Fully Quantum Approach

When both electrons and fields are quantized, the Hamiltonian includes three parts: photons, electrons, and interactions. The Hamiltonian is

$$\begin{aligned}\mathcal{H} &= \mathcal{H}_F + \mathcal{H}_e + \mathcal{H}_I && (4.1)\\ &= \sum_m \hbar\omega_m a_m^\dagger a_m + \sum_n E_n |E_n\rangle\langle E_n| - \mathbf{E}\cdot\mathbf{d}. && (4.2)\end{aligned}$$

It should be noted that both the field $\mathbf{E}$ and the dipole $\mathbf{d}$ are operators. The electric field operator is

$$\mathbf{E} = \sum_m \frac{\mathcal{E}_m a_m + \mathcal{E}_m^* a_m^\dagger}{2}, \tag{4.3}$$

and the dipole matrix operator in the energy basis is

$$\begin{pmatrix} \mathbf{d}_{11} & \mathbf{d}_{12} & \cdots \\ \mathbf{d}_{21} & \mathbf{d}_{22} & \\ \vdots & & \ddots \end{pmatrix}, \tag{4.4}$$

with $\mathbf{d}_{nn'} = \langle E_n|\mathbf{d}|E_{n'}\rangle$ and $\mathbf{d} = q\mathbf{r} = -e\mathbf{r}$.

The Hilbert space of the Hamiltonian includes both the photon and electron parts. The total space is indeed the tensor direct product of each space,

$$|\psi\rangle = |\text{photon}\rangle \otimes |\text{electron}\rangle. \tag{4.5}$$

The dimension of the total space is the product of the dimension of each space. In this definition, the photonic operators such as a and $a^\dagger$ will only be applied on the photonic ket $|\text{photon}\rangle$, and the electronic operators such as $\mathbf{d}$ will only be applied on the electronic ket $|\text{electron}\rangle$.

$$\langle\psi|\mathcal{H}_F|\psi\rangle = \langle\text{photon}|\mathcal{H}_F|\text{photon}\rangle \otimes \langle\text{electron}|\text{electron}\rangle = \langle\text{photon}|\mathcal{H}_F|\text{photon}\rangle \otimes \mathbb{1}_e, \tag{4.6}$$

$$\langle\psi|\mathcal{H}_e|\psi\rangle = \langle\text{photon}|\text{photon}\rangle \otimes \langle\text{electron}|\mathcal{H}_e|\text{electron}\rangle = \mathbb{1}_F \otimes \langle\text{electron}|\mathcal{H}_e|\text{electron}\rangle, \tag{4.7}$$

$$\langle\psi|\mathbf{E}\cdot\mathbf{d}|\psi\rangle = \langle\text{photon}|\mathbf{E}|\text{photon}\rangle \cdot \langle\text{electron}|\mathbf{d}|\text{electron}\rangle. \tag{4.8}$$

For example, we can write the photonic ket on a number basis and the electron ket in the energy basis,

$$|\text{photon}\rangle = \sum_n C_n|n\rangle, \tag{4.9}$$

$$|\text{electron}\rangle = \sum_m D_m|E_m\rangle. \tag{4.10}$$

Now, all the possible states of the total space can be written as

$$|\psi\rangle = \left(\sum_n C_n|n\rangle\right) \otimes \left(\sum_m D_m|E_m\rangle\right). \tag{4.11}$$

In principle, the dimension of the total space is infinite since the dimension of the number state is infinite. In practical computation, we will truncate the photon number so that the maximum number is finite, say n_m. The photon basis vectors now include $|0\rangle$, $|1\rangle$,... , $|n_m\rangle$, so the dimension of the photonic part is m. If we now consider a two-level system of electrons, the dimension of the total space is $m \times 2$. All the basis vectors of the total space are $|0\rangle|E_c\rangle$, $|1\rangle|E_c\rangle$,... , $|n_m\rangle|E_c\rangle$, and $|0\rangle|E_v\rangle$, $|1\rangle|E_v\rangle$,... , $|n_m\rangle|E_v\rangle$.

4.1 Two-Level System and Single-Mode Photons

The Hamiltonian is

$$\mathcal{H} = \hbar\omega a^\dagger a + \begin{pmatrix} E_c & 0 \\ 0 & E_v \end{pmatrix} - \mathbf{E}\cdot\mathbf{d}. \tag{4.12}$$

where the electric field operator is

$$\mathbf{E} = \frac{\boldsymbol{\mathcal{E}}_\omega a + \boldsymbol{\mathcal{E}}_\omega^* a^\dagger}{2}, \tag{4.13}$$

and the dipole matrix operator is

$$\begin{pmatrix} 0 & \mathbf{d}_{cv} \\ \mathbf{d}_{vc} & 0 \end{pmatrix}, \tag{4.14}$$

where we assume that the diagonal terms vanish. The transition rate from $|n\rangle|E_c\rangle$ to $|n+1\rangle|E_v\rangle$ is obtained by

$$W_{\text{emission}} = \frac{\pi}{2}\frac{|\langle n+1|\langle E_v|\mathbf{H}_I|n\rangle|E_c\rangle|^2}{\hbar^2}\delta(\omega-\omega_{cv}) \tag{4.15}$$

$$= \frac{(n+1)\pi}{2}\frac{|\boldsymbol{\mathcal{E}}_\omega\cdot\mathbf{d}_{cv}|^2}{\hbar^2}\delta(\omega-\omega_{cv}). \tag{4.16}$$

An interesting result occurs when $n = 0$. The emission is not zero when $n = 0$. This is the phenomenon of "spontaneous emission". When $n > 0$, it corresponds to the stimulated emission. The transition rate from $|n\rangle|E_v\rangle$ to $|n-1\rangle|E_c\rangle$ is obtained by

$$W_{\text{absorption}} = \frac{\pi}{2}\frac{|\langle n-1|\langle E_c|\mathbf{H}_I|n\rangle|E_v\rangle|^2}{\hbar^2}\delta(\omega-\omega_{cv}) \tag{4.17}$$

$$= \frac{n\pi}{2}\frac{|\boldsymbol{\mathcal{E}}_\omega^*\cdot\mathbf{d}_{vc}|^2}{\hbar^2}\delta(\omega-\omega_{cv}). \tag{4.18}$$

4.2 Jaynes-Cummings Model

The TLS and single-mode photon Hamiltonian can be further simplified with the RWA,

The Hamiltonian is

$$\mathcal{H} = \hbar\omega a^\dagger a + \begin{pmatrix} E_c & 0 \\ 0 & E_v \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & \boldsymbol{\mathcal{E}}_\omega \cdot \mathbf{d}_{cv} a + \boldsymbol{\mathcal{E}}_\omega^* \cdot \mathbf{d}_{cv} a^\dagger \\ \boldsymbol{\mathcal{E}}_\omega \cdot \mathbf{d}_{vc} a + \boldsymbol{\mathcal{E}}_\omega^* \cdot \mathbf{d}_{vc} a^\dagger & 0 \end{pmatrix} \tag{4.19}$$

$$\simeq \hbar\omega a^\dagger a + \begin{pmatrix} E_c & 0 \\ 0 & E_v \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & \boldsymbol{\mathcal{E}}_\omega \cdot \mathbf{d}_{cv} a \\ \boldsymbol{\mathcal{E}}_\omega^* \cdot \mathbf{d}_{vc} a & 0 \end{pmatrix} \tag{4.20}$$

$$= \hbar\omega a^\dagger a + \frac{E_c + E_v}{2} + \frac{\hbar\omega_{cv}}{2}\sigma_z + \hbar\left(\lambda\sigma_+ a + \lambda^* \sigma_- a^\dagger\right) \tag{4.21}$$

where

$$\lambda = \frac{-\boldsymbol{\mathcal{E}}_\omega \cdot \mathbf{d}_{cv}}{2\hbar}. \tag{4.22}$$

The average energy $\frac{E_c+E_v}{2}$ is only a constant, so as irrelevant to dynamics. In most cases, it is possible to make λ real by setting the phase of $\mathbf{d}_{cv}$. The Jaynes-Cummings Model is then obtained as

$$\mathcal{H}_{JC} = \hbar\omega a^\dagger a + \frac{\hbar\omega_{cv}}{2}\sigma_z + \hbar\lambda\left(\sigma_+ a + \sigma_- a^\dagger\right). \tag{4.23}$$

We have used the Pauli matrices

$$\sigma_z = |E_c\rangle\langle E_c| - |E_v\rangle\langle E_v| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.24}$$

$$\sigma_+ = |E_c\rangle\langle E_v| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{4.25}$$

$$\sigma_- = |E_v\rangle\langle E_c| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4.26}$$

The electron number operator is an identity,

$$N_e = |E_c\rangle\langle E_c| + |E_v\rangle\langle E_v|, \tag{4.27}$$

and the excitation number operator is

$$N_{ex} = |E_c\rangle\langle E_c| + a^\dagger a. \tag{4.28}$$

These numbers are conservative since the commutators vanish

$$[\mathcal{H}, N_e] = 0, \tag{4.29}$$

$$[\mathcal{H}, N_{ex}] = 0, \tag{4.30}$$

which mean that the total Hamiltonian can be **block-diagonalized**, and in each block, the excitation number and the electron number are the same. The basis kets are

$$|n\rangle \otimes |E_m\rangle \equiv |n\rangle|E_m\rangle \tag{4.31}$$

where $E_m = E_c$ or E_v and $n = 0,\ 1,\ ,2,\ 3,...$ It seems that if we want to use the number states as the basis, the dimension of the Hamiltonian would be infinite. This is true, but the Hamiltonian can be block-diagonalized. **Because the excitation number is conserved, only the states with the same excitation number are coupled.** Within each block, the excitation number is the same. Eventually, one finds that each block is just a 2 by 2 matrix. This is because the state $|E_c\rangle|n\rangle$ is only coupled to $|E_v\rangle|n+1\rangle$. The problem is then solved using a two-dimensional Hamiltonian since each block is independent.

Exercise 1: Excitation Number

Show Eq. (4.30).

The Hamiltonian is decomposed as

$$\mathcal{H}_{JC} = \mathcal{H}_N + \mathcal{H}_D \tag{4.32}$$

$$\mathcal{H}_N = \hbar\omega N_{ex} - \hbar\frac{\omega}{2}N_e, \tag{4.33}$$

$$\mathcal{H}_D = -\frac{\hbar\Delta}{2}\sigma_z + \hbar\lambda\left(\sigma_+ a + \sigma_- a^\dagger\right). \tag{4.34}$$

with $\omega = \omega_{cv} + \Delta$. The two Hamiltonians $\mathcal{H}_N$ and $\mathcal{H}_D$ commute with each other,

$$[\mathcal{H}_N, \mathcal{H}_D] = 0, \tag{4.35}$$

which means the two Hamiltonians are decoupled, so

$$e^{-i\frac{\mathcal{H}_N+\mathcal{H}_D}{\hbar}t} = e^{-i\frac{\mathcal{H}_N}{\hbar}t}e^{-i\frac{\mathcal{H}_D}{\hbar}t} = e^{-i\frac{\mathcal{H}_D}{\hbar}t}e^{-i\frac{\mathcal{H}_N}{\hbar}t}. \tag{4.36}$$

On the basis of Eq. (4.31), the Hamiltonian $\mathcal{H}_N$ is indeed diagonal, which means that as time increases, $\mathcal{H}_N$ only adds the phase in each basis vector but does not cause the transitions between the basis kets. The physical reason is that the Hamiltonian $\mathcal{H}_N$ describes the conservative numbers so that it is irrelevant to dynamics. Therefore, the dynamics is given by $\mathcal{H}_D$. We can use the interaction picture where $\mathcal{H}_0 = \mathcal{H}_D$ so that the dynamics is given by

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle_I = \mathcal{H}_D|\psi\rangle_I. \tag{4.37}$$

The ket here is in the interaction picture. Because of being block-diagonalized, the dimension of $|\psi\rangle_I$ is effectively 2.

Example 1: Number State

Let the light in the number state $|n\rangle$. The two basis kets are

$$|n+1\rangle|E_v\rangle \equiv |i\rangle, \tag{4.38}$$

$$|n\rangle|E_c\rangle \equiv |f\rangle. \tag{4.39}$$

An arbitrary state in the interaction picture is

$$|\psi(t)\rangle = C_i(t)|i\rangle + C_f(t)|f\rangle. \tag{4.40}$$

Plugging this state in Eq. (4.37), we obtain

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}C_f\\C_i\end{pmatrix}=\begin{pmatrix}-\frac{\hbar\Delta}{2} & \sqrt{n+1}\hbar\lambda\\ \sqrt{n+1}\hbar\lambda & \frac{\hbar\Delta}{2}\end{pmatrix}\begin{pmatrix}C_f\\C_i\end{pmatrix}. \tag{4.41}$$

The eigenfrequencies are

$$\omega_\pm=\pm\sqrt{\frac{\Delta^2}{4}+(n+1)\lambda^2}. \tag{4.42}$$

and the eigenvectors (using the Bloch sphere representation) are

$$|\omega_+\rangle=\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}e^{-i\omega_+t} \tag{4.43}$$

$$|\omega_-\rangle=\begin{pmatrix}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}\end{pmatrix}e^{-i\omega_-t} \tag{4.44}$$

with

$$\theta=-\tan^{-1}\left(\frac{2\sqrt{n+1}\lambda}{\Delta}\right). \tag{4.45}$$

If the initial state is $C_i=1$ and $C_f=0$, the solution becomes

$$|\psi\rangle=\sin\frac{\theta}{2}|\omega_+\rangle-\cos\frac{\theta}{2}|\omega_-\rangle, \tag{4.46}$$

$$C_i(t)=\cos\omega_+t+i\cos\theta\sin\omega_+t, \tag{4.47}$$

$$C_f(t)=-i\sin\theta\sin\omega_+t. \tag{4.48}$$

The population of the excited state $n_e=|C_f(t)|^2$ is

$$n_e=\sin^2\theta\sin^2\omega_+t, \tag{4.49}$$

$$=\sin^2\theta\sin^2\sqrt{\frac{\Delta^2}{4}+(n+1)\lambda^2}t. \tag{4.50}$$

This is the Rabi oscillation between the states $|E_v\rangle|n+1\rangle$ and $|E_c\rangle|n\rangle$. Only when the detuning is zero, we have $\sin\theta=1$ and the maximum excitation. The Rabi frequency is

$$\omega_+=\sqrt{\frac{\Delta^2}{4}+(n+1)\lambda^2}. \tag{4.51}$$

The Rabi frequency depends on the number of photons. One novel case is $n=0$ where the frequency is not zero but

$$\omega_+(n=0)=\sqrt{\frac{\Delta^2}{4}+\lambda^2}. \tag{4.52}$$

This means that there exists the Rabi oscillation even when there is no photon.[a] This is called the “vacuum Rabi oscillations”.

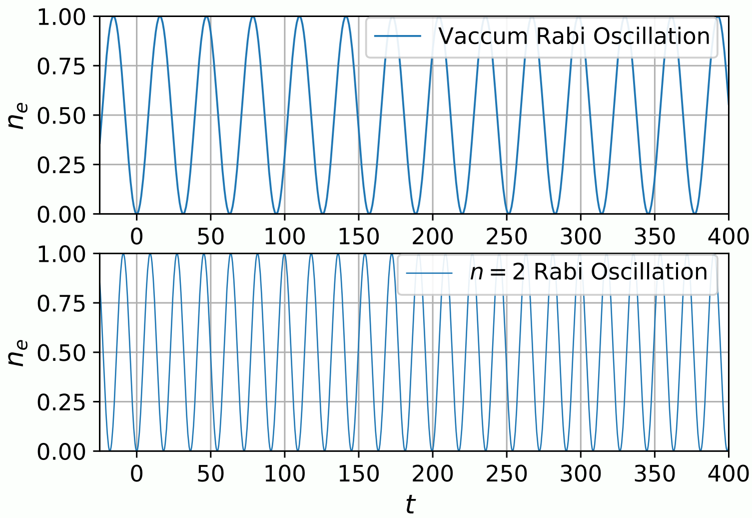

Figure 4: Rabi oscillations of the JC models for $n = 0$ and $n = 2$. The other parameters are $\Delta = 0$ and $\lambda = 0.1$

.

[a]Though, the vacuum energy is nonzero!

4.3 JC models with a Coherent State

Let us consider a more general situation where the photon state is

$$|\text{field}\rangle = \sum_{n=0}^{\infty} C_n |n\rangle, \tag{4.53}$$

and the two-level system is

$$|\text{TLS}\rangle = C_c |E_c\rangle + C_v |E_v\rangle. \tag{4.54}$$

The total state is

$$|\psi\rangle = |\text{field}\rangle \otimes |\text{TLS}\rangle. \tag{4.55}$$

The solution is then (when $\Delta = 0$)

$$|\psi\rangle = \sum_n [C_c C_n \cos(\omega_{n+1} t) - i C_v C_{n+1} \sin(\omega_{n+1} t)] |n\rangle |E_c\rangle \tag{4.56}$$

$$+ \sum_n [C_v C_{n+1} \cos(\omega_{n+1} t) - i C_c C_n \sin(\omega_{n+1} t)] |n+1\rangle |E_v\rangle, \tag{4.57}$$

where

$$\omega_n = \omega_+(n). \tag{4.58}$$

Let the initial state be $C_c = 0$ and $C_v = 1$. The population of the excited state is

$$n_e = |C_c(t)|^2 = \sum_n |C_{n+1}|^2 \sin^2 \omega_{n+1} t \tag{4.59}$$

$$= \sum_n |C_{n+1}|^2 \left(\frac{1 - \cos 2\omega_{n+1} t}{2} \right) \tag{4.60}$$

$$= \frac{1}{2} - \sum_n |C_{n+1}|^2 \left(\frac{\cos 2\omega_{n+1} t}{2} \right). \tag{4.61}$$

In terms of n, we obtain

$$n_e = \frac{1}{2} - \sum_n |C_{n+1}|^2 \left(\frac{\cos 2\lambda\sqrt{n+1}\, t}{2} \right). \tag{4.62}$$

Figure 5 shows the populations in the cases of coherent states. Even with a coherent state, the population is not a simple harmonic oscillation as in the classical case. There are two new properties. First, the oscillation lasts for a time τ_c (the duration of the wave packet.) and **collapses**. It is shown that the time τ_c is in the limit $n \to \infty$,

$$\tau_c \simeq \frac{\sqrt{2}}{\lambda}. \tag{4.63}$$

After a rephasing time τ_{rp}, the oscillation comes back. This is called the **revival**. The time τ_{rp} is in the limit $n \to \infty$,

$$\tau_{\mathrm{rp}} \simeq \frac{4\pi|\alpha|}{\lambda}. \tag{4.64}$$

Two properties of the JC model are

- Collapsing
- Revival

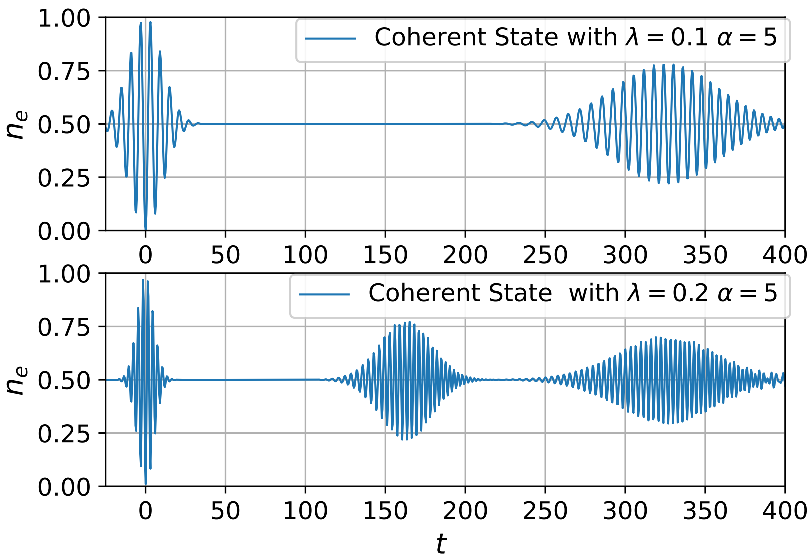

Figure 5: Rabi oscillations of the JC models for a coherent state. Collapsing and revival appear.

4.4 Dressed States (Strong-Coupling Regime)

We focused on the dynamics of the JC model. Now, we discuss the eigenstates of the JC model. First, the photon energy in the vacuum is $E = n\hbar\omega$.[7] In a cavity, photons are coupled with the TLS. As a result, the photon energies are shifted. We can think that the combination of photons and the TLS leads to a new state called the "dressed state", or in the context of condensed matter physics, "polaritons". The JC Hamiltonian is block-diagonalized. Each block, denoted as $\mathcal{H}^{(n)}$, is a 2 by 2 matrix,

$$\mathcal{H}^{(n)} = n\hbar\omega + \begin{pmatrix} -\frac{\hbar\Delta}{2} & \sqrt{n+1}\hbar\lambda \\ \sqrt{n+1}\hbar\lambda & \frac{\hbar\Delta}{2} \end{pmatrix}, \tag{4.65}$$

where the matrix is spanned by the basis vectors from Eqs. (4.38) and (4.39). The eigenvalues are

$$E_{1n} = n\hbar\omega + \hbar\omega_n, \tag{4.66}$$
$$E_{2n} = n\hbar\omega - \hbar\omega_n, \tag{4.67}$$

where $\omega_n = \sqrt{\frac{\Delta^2}{4} + (n+1)\lambda^2}$ and the eigenvectors (using the Bloch sphere representation) are

$$|1n\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} e^{-i\omega_+ t} \tag{4.68}$$
$$|2n\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix} e^{-i\omega_- t} \tag{4.69}$$

with

$$\theta = -\tan^{-1}\left(\frac{2\sqrt{n+1}\lambda}{\Delta}\right). \tag{4.70}$$

The dressed photons are the eigenstates of the total system. Compared to photons in a vacuum, their frequencies shift and become non-degenerate. The splitting of dressed states is the origin of the Mollow triplet emissions.

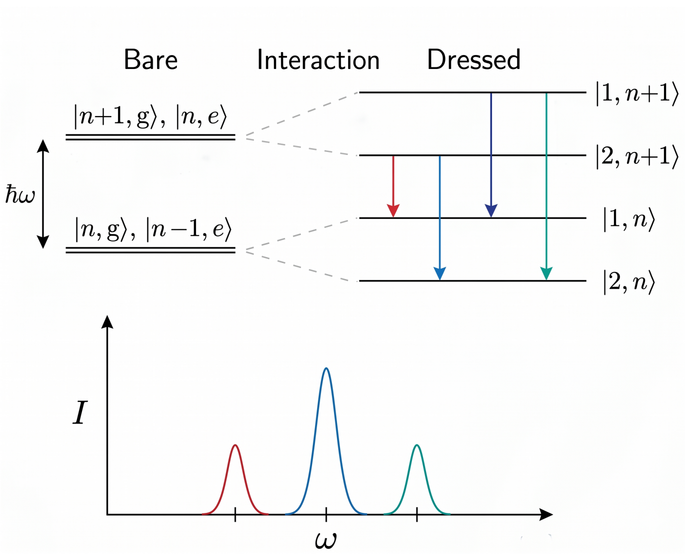

Figure 6: Mollow triplet emissions.

[7]We drop $1/2\hbar\omega$.

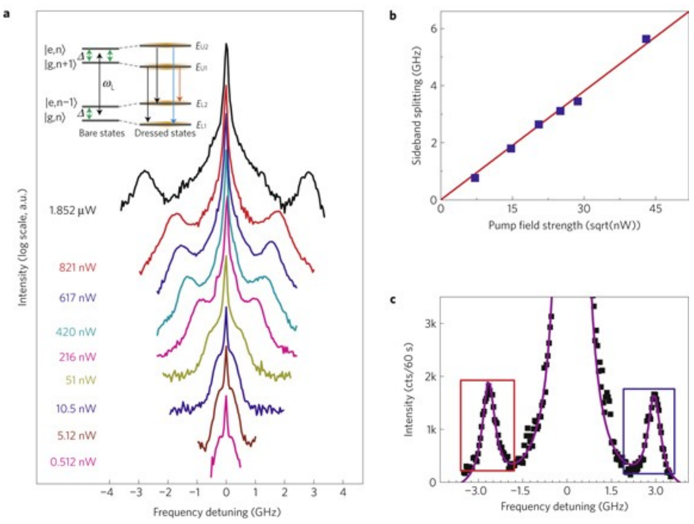

Figure 7: Experimental observation of the Mollow triplet emissions[2].

5 Outlook: Quantum Optics in Current Research

The mathematical frameworks derived in these notesfrom the semi-classical Rabi model to the fully quantized Jaynes-Cummings Hamiltonianare not merely historical milestones. They are the operational blueprints for the "Second Quantum Revolution." The ability to coherently control light-matter interactions at the single-photon and single-atom levels has given rise to entirely new fields of technology.

5.1 From Single Atoms to Quantum Circuits (Circuit QED)

Historically, testing the fully quantum approach required trapping individual atoms in ultra-high-vacuum optical cavities (Cavity QED). Today, the frontier has expanded to "Circuit QED." Researchers engineer macroscopic artificial atoms (such as superconducting transmons) and couple them to microwave transmission line resonators. Remarkably, the Jaynes-Cummings Hamiltonian governs these macroscopic superconducting circuits. Furthermore, the dressed-state physics and dispersive shifts discussed in this text are used daily in quantum computing laboratories to perform high-fidelity readouts of qubit states.

5.2 Quantum Computing and Simulation

The coherent exchange of energy between fields and matter is the fundamental mechanism by which quantum logic gates are executed. Precise control over the Rabi frequency (Ω_R) and the detuning (Δ) allows physicists to drive specific atomic transitions and entangle multiple qubits. Beyond universal computation, highly controlled light-matter interfaces are actively used as quantum simulators to model complex, strongly correlated many-body systems that are computationally intractable for classical supercomputers.

5.3 Quantum Communication and Information Networks

The realization of a “quantum internet” relies on distributing quantum entanglement over vast distances. This requires a reliable interface between “flying qubits” (traveling optical photons) and “stationary qubits” (atoms, trapped ions, or solid-state defects). Storing a delicate quantum state into a long-lived matter excitationand later retrieving it on demanddemands an absolute mastery of the fully quantum treatment of spontaneous and stimulated emission pathways discussed in Section 4.

5.4 Quantum Metrology and Sensing

Classical measurement precision is fundamentally limited by the shot-noise limit of standard laser light, which we modeled as the generation of coherent states via classical macroscopic currents. By engineering deep, strong-coupling light-matter interactions, researchers can now generate highly non-classical states of light, such as squeezed states or macroscopically entangled states. These quantum resources enable precision sensing well beyond the standard quantum limit, enabling breakthroughs ranging from the detection of infinitesimal gravitational waves to the mapping of magnetic fields within living biological tissues.

References

[1] J. J. Sakurai, *Modern Quantum Mechanics*, 1994 and 2010

[2] Nick Vamivakas, A., et al. "Spin-resolved quantum-dot resonance fluorescence." Nature Physics 5.3 (2009): 198-202.