Optical Components and Measurements

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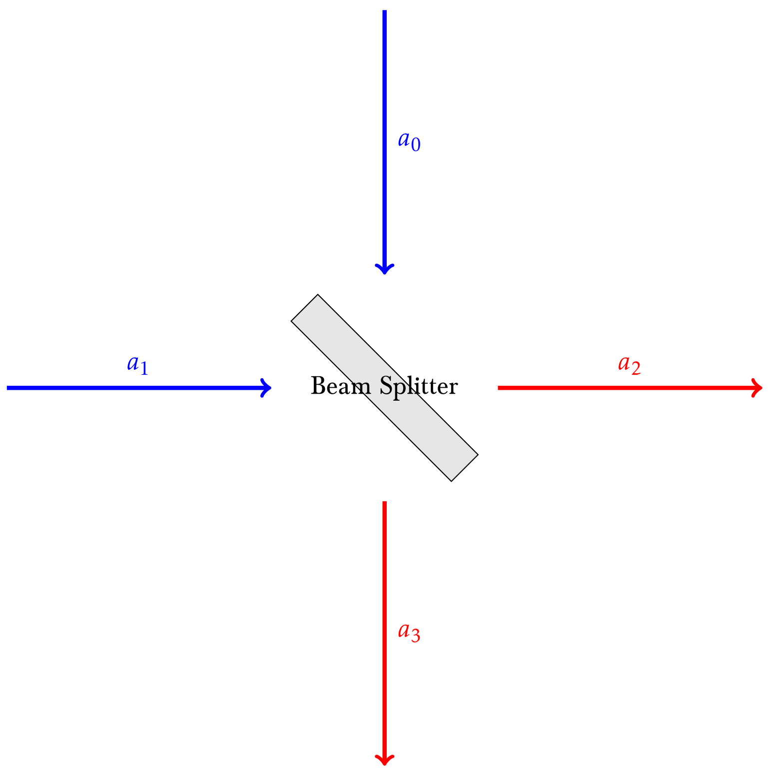
Contents

Beam splitters are essential optical components that have a profound impact on both classical and quantum optics by dividing incoming light beams into reflected and transmitted parts. This note delves into their quantum mechanical behavior, illustrating how they manipulate light at the photon level. We begin by examining the classical description, where amplitudes split predictably, and then transition to quantum descriptions using annihilation operators to uncover uniquely quantum phenomena.

A key concept explored is the transformation of single-photon states, which leads to entangled outputs. This is notably demonstrated in the Mach-Zehnder interferometer, where quantum interference allows for precise measurements such as phase determination, with implications for quantum computing and sensing. We also investigate the Hong-Ou-Mandel effect, a striking example of quantum interference where two photons incident on a symmetric beam splitter exhibit "bunching" rather than the classically expected distribution. This non-classical result is fundamental to linear optical quantum computing. Furthermore, the note differentiates the outcomes for N-photon states, which become entangled after passing through a beam splitter, from coherent states, which remain unentangled.



- Indistinguishablity of boson

- Gaussian boson sampling

- Here is a link to a YouTube video about quantum optics and Gaussian sampling with a billiard picture: Gaussian wave packet in a quantum ellipse billiard.

1 Beam Splitters

A beam splitter is an optical component which is partially transparent. An incident beam on a beam splitter is partially reflected and partially transmitted and thus split into two beams. Classically, an incident beam with an amplitude A_1 is split into a reflected beam with the amplitude A_1 and a transmitted beam with the amplitude A_2. The amplitudes are related by the coefficients of reflection and transmission,

$$A_2 = rA_1, \tag{1.1}$$
$$A_3 = tA_1. \tag{1.2}$$

We can also consider another case where a beam of the amplitude A_0 is incident on the other side of the beam splitter. In this case, the amplitudes are related by

$$A_2 = t'A_0, \tag{1.3}$$
$$A_3 = r'A_0. \tag{1.4}$$

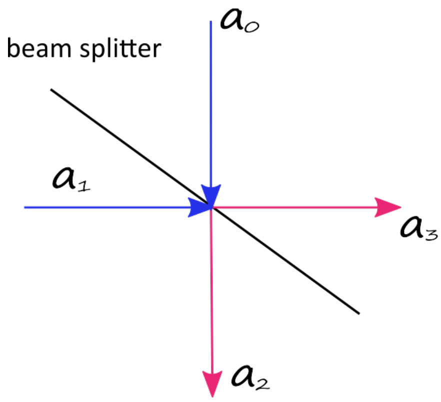


Figure 1: Beam splitter. Quantum descriptions where the annihilation operators replace the amplitudes.

The scattering matrix describes a beam splitter

$$\begin{pmatrix} A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} \equiv U \begin{pmatrix} A_0 \\ A_1 \end{pmatrix}. \tag{1.5}$$

Because of energy conservation, the matrix U must be unitary so that

$$|r|^2 + |t|^2 = 1 \tag{1.6}$$
$$|r'|^2 + |t'|^2 = 1 \tag{1.7}$$
$$tr^* + t'^* r' = 0. \tag{1.8}$$

In order to satisfy Eq. (1.8), the phase of each beam cannot be arbitrary. Let's consider the example of a symmetric beam splitter. Let the phase between r and t is θ , that is, $\mathrm{Arg}\left[\frac{t}{r}\right] = \theta$. A symmetric beam splitter implies $\mathrm{Arg}\left[\frac{t'}{r'}\right] = \theta$. Eq. (1.8) becomes

$$|r||t|\left(e^{i\theta} - e^{-i\theta}\right) = 0, \tag{1.9}$$

which gives $\theta = \pm\frac{\pi}{2}$. A discussion about the phases can be found in the Ref. [1].

The quantum description of a beam splitter is to replace amplitudes with annihilation operators. Let the right-going photons have the annihilation operator a_1, and the bottom-going photons have the annihilation operator a_0. The scattering matrix describes a beam splitter

$$\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = U \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}, \tag{1.10}$$

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} t'^* & r' \\ r^* & t^* \end{pmatrix} \begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = U^\dagger \begin{pmatrix} a_2 \\ a_3 \end{pmatrix}. \tag{1.11}$$

The incident beams, a_0 and a_1, are treated as two independent modes with the commutation relations

$$[a_0, a_0^\dagger] = 1, \tag{1.12}$$

$$[a_1, a_1^\dagger] = 1, \tag{1.13}$$

$$[a_0, a_1^\dagger] = 0. \tag{1.14}$$

From Eqs. (1.12), (1.13), (1.14), and (1.10), one can show that the operators a_2 and a_3 automatically satisfy

$$[a_i, a_j^\dagger] = \delta_{ij}. \tag{1.15}$$

Physically, these relations mean that after a beam splitter, a beam is split into two independent modes a_2 and a_3.

Below, we will discuss what happens to a quantum light after passing a beam splitter. We will consider the cases of a single photon state, N-photon state, and a coherent state. We will see that the Fock states exhibit quantum natures, where the output states are entangled, while the output state of a coherent state can be factorized.

1.1 Single Photon

The incident state $|0\rangle_0|1\rangle_1$ can be expressed as

$$|0\rangle_0|1\rangle_1 = a_1^\dagger|0\rangle_0|0\rangle_1. \tag{1.16}$$

From Eq. (1.11), the creation operators $a_0^\dagger$ and $a_1^\dagger$ are related to $a_2^\dagger$ and $a_3^\dagger$ by

$$\begin{pmatrix} a_0^\dagger \\ a_1^\dagger \end{pmatrix} = U^T \begin{pmatrix} a_2^\dagger \\ a_3^\dagger \end{pmatrix}. \tag{1.17}$$

After incidence on the beam splitter, the state becomes

$$U^T a_1^\dagger|0\rangle_0|0\rangle_1 = (ra_2^\dagger + ta_3^\dagger)|0\rangle_2|0\rangle_3 \tag{1.18}$$

$$= r|1\rangle_2|0\rangle_3 + t|0\rangle_2|1\rangle_3, \tag{1.19}$$

which is an entangled state.

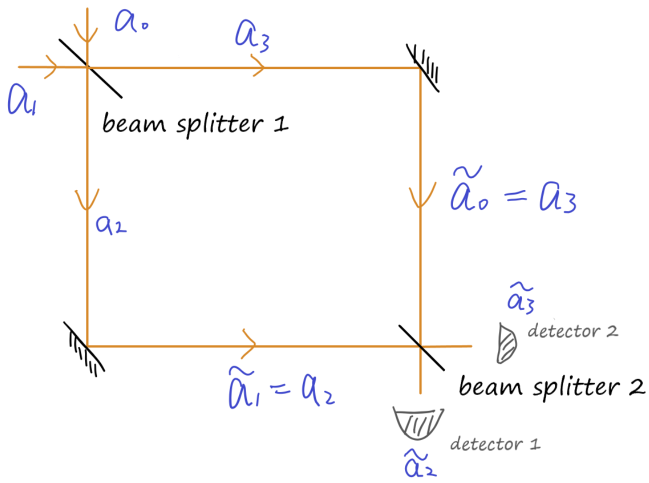


Figure 2: MachZehnder interferometer.

Consider a MachZehnder interferometer with two 50:50 beam splitters of $r = \frac{i}{\sqrt{2}}$ and $t = \frac{1}{\sqrt{2}}$. Let the initial state be $|0\rangle_0|1\rangle_1$. After the first beam splitter, the state becomes

$$\frac{i}{\sqrt{2}}|1\rangle_2|0\rangle_3 + \frac{1}{\sqrt{2}}|0\rangle_2|1\rangle_3. \tag{1.20}$$

When the state arrives at the second beam splitter, the state becomes

$$\frac{i}{\sqrt{2}}|1\rangle_2|0\rangle_3 + \frac{e^{i\theta}}{\sqrt{2}}|0\rangle_2|1\rangle_3, \tag{1.21}$$

where θ is a phase shift due to the difference of the two paths.

To deal with the second beam splitter, we first rename the modes. Modes 2 (a_2) and 3 (a_3) in the Eq. (1.21) become the incident beams to the second beam splitter. We rename Mode 2 (a_2) as the new Mode 1 ($\tilde{a}_1$) since it is right-going and Mode 3 (a_3) as the new Mode 0 ($\tilde{a}_0$) since it is bottom-going. For simplicity, we skip the tilde signs below. Now, we can use the same scattering matrix to find the final state after the second beam splitter,

$$\begin{pmatrix} a_2^\dagger \\ a_3^\dagger \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{e^{i\theta}}{\sqrt{2}} a_0^\dagger \\ \frac{i}{\sqrt{2}} a_1^\dagger \end{pmatrix} \tag{1.22}$$

$$= \begin{pmatrix} \frac{(e^{i\theta}-1)}{2} a_0^\dagger \\ \frac{i(e^{i\theta}+1)}{2} a_1^\dagger \end{pmatrix} \tag{1.23}$$

The probability at D_1 is $\left|\frac{(e^{i\theta}-1)}{2}\right|^2 = \sin^2\frac{\theta}{2}$. The probability at D_2 is $\left|\frac{i(e^{i\theta}+1)}{2}\right|^2 = \cos^2\frac{\theta}{2}$. This is a more rigorous description of single-photon interference.

1.2 Two-Photon Hong-Ou-Mandel Effect

The Hong-Ou-Mandel effect is a simple but novel phenomenon of quantum optics. It is an example where quantum interference leads to a non-classical result. Consider a symmetric beam 50-50 beam splitter. Now, two single-photon states hit the beam splitter from different sides. As shown in Fig. 3, the initial state is $|1\rangle_0|1\rangle_1$. What will be the output state? Since the total input photon number is two, the output state could be $|2\rangle_2|0\rangle_3$, $|0\rangle_2|2\rangle_3$, and $|1\rangle_2|1\rangle_3$. Here, the subscripts of the ket denote on which side the photon state is, as in Fig. 3. Hence, a general expression for the output state is the superposition of the above possible states,

$$|\text{output}\rangle = \alpha|2\rangle_2|0\rangle_3 + \beta|0\rangle_2|2\rangle_3 + \gamma|1\rangle_2|1\rangle_3. \tag{1.24}$$

From the classical point of view, the state $|1\rangle_2|1\rangle_3$ seems the most possible state. However, in 1987, Hong et al. showed that for a symmetric beam splitter, $\gamma = 0$, that is, the probability of the state $|1\rangle_2|1\rangle_3$ is zero.[2] Specifically, the output state is

$$\begin{aligned}
|\text{output}\rangle &= U^T a_0^\dagger|0\rangle U^T a_1^\dagger|0\rangle && (1.25)\\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} a_0^\dagger \\ 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 \\ a_1^\dagger \end{pmatrix}|0\rangle && (1.26)\\
&= \frac{1}{2}\left(a_2^\dagger + ia_3^\dagger\right)\left(ia_2^\dagger + a_3^\dagger\right)|0\rangle && (1.27)\\
&= \frac{i}{2}\left[\left(a_2^\dagger\right)^2 + \left(a_3^\dagger\right)^2\right]|0\rangle && (1.28)\\
&= \frac{i}{\sqrt{2}}(|2\rangle_2|0\rangle_3 + |0\rangle_2|2\rangle_3) && (1.29)
\end{aligned}$$

Surprisingly, two identical single photons are bunched after hitting a beam splitter from different sides. In the linear optical quantum computing, the Hong-Ou-Mandel effect is the mechanism for the logic gates.

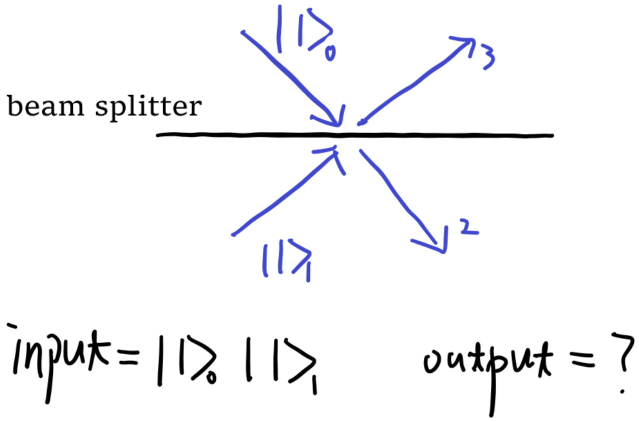


Figure 3: Hong-Ou-Mandel effect.

1.3 N-Photon

Let the initial state be $|0\rangle_0|N\rangle_1 = \frac{(a_1^\dagger)^N}{\sqrt{N!}}|0\rangle$. After a beam splitter, the state becomes

$$\frac{(Ua_1^\dagger)^N}{\sqrt{N!}}|0\rangle = \frac{(ta_2^\dagger + ra_3^\dagger)^N}{\sqrt{N!}}|0\rangle. \tag{1.30}$$

1.4 Coherent States

Let the initial state be $|0\rangle_0|\alpha\rangle_1 = D_1[\alpha]|0\rangle = e^{\alpha a_1^\dagger - \alpha^* a_1}|0\rangle$. After a beam splitter, the state becomes

$$e^{\alpha U a_1^\dagger - \alpha^* U^\dagger a_1}|0\rangle = e^{\alpha t a_2^\dagger - \alpha^* t^* a_2} e^{\alpha r a_3^\dagger - \alpha^* r^* a_3}|0\rangle \tag{1.31}$$

$$= D_2[t\alpha]D_3[r\alpha]|0\rangle \tag{1.32}$$

$$= |t\alpha\rangle_2|r\alpha\rangle_3. \tag{1.33}$$

The input of a coherent state is split into a product of two coherent states. Unlike the single-photon case, this state is not entangled.

Consider a MachZehnder interferometer with two 50:50 beam splitters of $r = \frac{i}{\sqrt{2}}$ and $t = \frac{1}{\sqrt{2}}$. Let the initial state be $|0\rangle_0|\alpha\rangle_1$. After the first beam splitter, the state becomes

$$\left|\frac{i\alpha}{\sqrt{2}}\right\rangle_2 \left|\frac{\alpha}{\sqrt{2}}\right\rangle_3. \tag{1.34}$$

When the state arrives at the second beam splitter, the state becomes

$$\left|\frac{i\alpha}{\sqrt{2}}\right\rangle_2 \left|\frac{e^{i\theta}\alpha}{\sqrt{2}}\right\rangle_3. \tag{1.35}$$

where θ is a phase shift due to the difference of the two paths. The final state after the second beam splitter is (see the figure: a_2 is the new a_1 and a_3 is the new a_0)

$$\left|\frac{\left(e^{i\theta} - 1\right)}{2}\right\rangle_2 \left|\frac{i\left(e^{i\theta} + 1\right)}{2}\right\rangle_3. \tag{1.36}$$

The intensity at D_1 is $|\alpha|^2\left|\frac{e^{i\theta}-1}{2}\right|^2 = \sin^2\frac{\theta}{2}|\alpha|^2$. The intensity at D_2 is $|\alpha|^2\left|\frac{e^{i\theta}+1}{2}\right|^2 = \cos^2\frac{\theta}{2}|\alpha|^2$. The two output beams are both coherent states. Thus, the phase θ can be obtained by

$$\frac{I_2 - I_1}{|\alpha|^2} = \cos\theta. \tag{1.37}$$

However, the amplitudes have uncertainty $\sigma(n) = \sqrt{\bar{n}}$. Thus, the phase has the uncertainty $\sigma(\theta) \sim \frac{1}{\sqrt{\bar{n}}}$. In experiments, we would like to use a coherent light (laser) with a well-defined phase and a strong intensity such that the uncertainty in the phase is small. However, a strong-intensity light may lead to more noise, such as radiation pressures, thermal noises, etc. To solve this dilemma, lights with small $\sigma(n)$ are used. These lights are non-classical lights.

References

[1] Z. Y. Ou and L. Mandel, Derivation of reciprocity relations for a beam splitter from energy balance, American Journal of Physics 57, 66 (1989)

[2] C. K. Hong; Z. Y. Ou & L. Mandel . "Measurement of subpicosecond time intervals between two photons by interference." Phys. Rev. Lett. 59 (18): 20442046 (1987)