Optical Components and Measurements

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1 Beam Splitters

A beam splitter is an optical component which is partially transparent. An incident beam on a beam splitter is partially reflected and partially transmitted and thus split into two beams. Classically, an incident beam with an amplitude A_1 is split into a reflected beam with the amplitude A_1 and a transmitted beam with the amplitude A_2. The amplitudes are related by the coefficients of reflection and transmission,

$$A_2 = rA_1, \tag{1.1}$$
$$A_3 = tA_1. \tag{1.2}$$

We can also consider another case where a beam of the amplitude A_0 is incident on the other side of the beam splitter. In this case, the amplitudes are related by

$$A_2 = t'A_0, \tag{1.3}$$
$$A_3 = r'A_0. \tag{1.4}$$

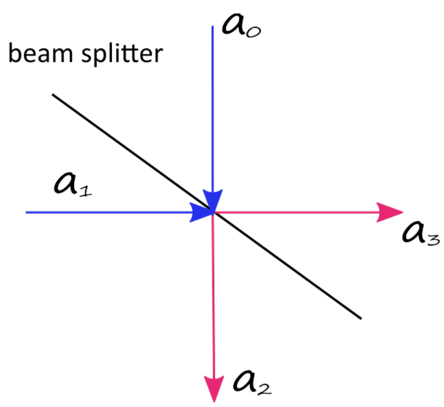

Figure 1: Beam splitter. Quantum descriptions where the annihilation operators replace the amplitudes.

The scattering matrix describes a beam splitter

$$\begin{pmatrix} A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} \equiv U \begin{pmatrix} A_0 \\ A_1 \end{pmatrix}. \tag{1.5}$$

Because of energy conservation, the matrix U must be unitary so that

$$|r|^2 + |t|^2 = 1 \tag{1.6}$$
$$|r'|^2 + |t'|^2 = 1 \tag{1.7}$$
$$tr^* + t'^* r' = 0. \tag{1.8}$$

In order to satisfy Eq. (1.8), the phase of each beam cannot be arbitrary. Let's consider the example of a symmetric beam splitter. Let the phase between r and t is θ , that is, $\mathrm{Arg}\left[\frac{t}{r}\right] = \theta$. A symmetric beam splitter implies $\mathrm{Arg}\left[\frac{t'}{r'}\right] = \theta$. Eq. (1.8) becomes

$$|r||t|\left(e^{i\theta} - e^{-i\theta}\right) = 0, \tag{1.9}$$

which gives $\theta = \pm\frac{\pi}{2}$. A discussion about the phases can be found in the Ref. [1].

The quantum description of a beam splitter is to replace amplitudes with annihilation operators. Let the right-going photons have the annihilation operator a_1, and the bottom-going photons have the annihilation operator a_0. The scattering matrix describes a beam splitter

$$\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = U \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}, \tag{1.10}$$

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} t'^* & r' \\ r^* & t^* \end{pmatrix} \begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = U^\dagger \begin{pmatrix} a_2 \\ a_3 \end{pmatrix}. \tag{1.11}$$

The incident beams, a_0 and a_1, are treated as two independent modes with the commutation relations

$$[a_0, a_0^\dagger] = 1, \tag{1.12}$$

$$[a_1, a_1^\dagger] = 1, \tag{1.13}$$

$$[a_0, a_1^\dagger] = 0. \tag{1.14}$$

From Eqs. (1.12), (1.13), (1.14), and (1.10), one can show that the operators a_2 and a_3 automatically satisfy

$$[a_i, a_j^\dagger] = \delta_{ij}. \tag{1.15}$$

Physically, these relations mean that after a beam splitter, a beam is split into two independent modes a_2 and a_3.

Below, we will discuss what happens to a quantum light after passing a beam splitter. We will consider the cases of a single photon state, N-photon state, and a coherent state. We will see that the Fock states exhibit quantum natures, where the output states are entangled, while the output state of a coherent state can be factorized.

1.1 Single Photon

The incident state $|0\rangle_0|1\rangle_1$ can be expressed as

$$|0\rangle_0|1\rangle_1 = a_1^\dagger|0\rangle_0|0\rangle_1. \tag{1.16}$$

From Eq. (1.11), the creation operators $a_0^\dagger$ and $a_1^\dagger$ are related to $a_2^\dagger$ and $a_3^\dagger$ by

$$\begin{pmatrix} a_0^\dagger \\ a_1^\dagger \end{pmatrix} = U^T \begin{pmatrix} a_2^\dagger \\ a_3^\dagger \end{pmatrix}. \tag{1.17}$$

After incidence on the beam splitter, the state becomes

$$U^T a_1^\dagger|0\rangle_0|0\rangle_1 = (ra_2^\dagger + ta_3^\dagger)|0\rangle_2|0\rangle_3 \tag{1.18}$$

$$= r|1\rangle_2|0\rangle_3 + t|0\rangle_2|1\rangle_3, \tag{1.19}$$

which is an entangled state.

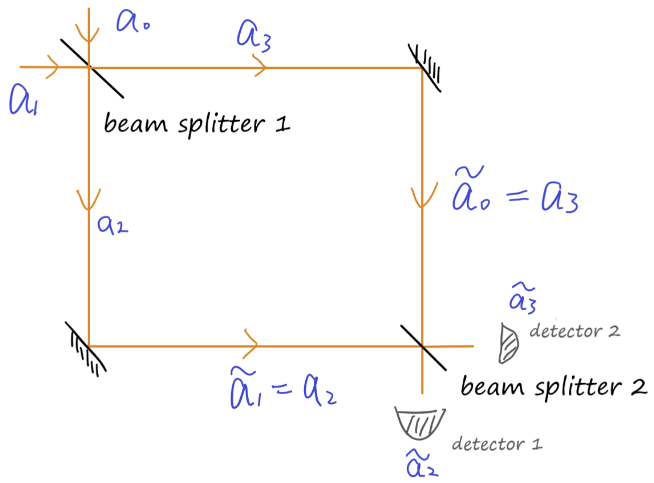

Figure 2: Mach–Zehnder interferometer.

Consider a Mach–Zehnder interferometer with two 50:50 beam splitters of $r = \frac{i}{\sqrt{2}}$ and $t = \frac{1}{\sqrt{2}}$. Let the initial state be $|0\rangle_0|1\rangle_1$. After the first beam splitter, the state becomes

$$\frac{i}{\sqrt{2}}|1\rangle_2|0\rangle_3 + \frac{1}{\sqrt{2}}|0\rangle_2|1\rangle_3. \tag{1.20}$$

When the state arrives at the second beam splitter, the state becomes

$$\frac{i}{\sqrt{2}}|1\rangle_2|0\rangle_3 + \frac{e^{i\theta}}{\sqrt{2}}|0\rangle_2|1\rangle_3, \tag{1.21}$$

where θ is a phase shift due to the difference of the two paths.

To deal with the second beam splitter, we first rename the modes. Modes 2 (a_2) and 3 (a_3) in the Eq. (1.21) become the incident beams to the second beam splitter. We rename Mode 2 (a_2) as the new Mode 1 ($\tilde{a}_1$) since it is right-going and Mode 3 (a_3) as the new Mode 0 ($\tilde{a}_0$) since it is bottom-going. For simplicity, we skip the tilde signs below. Now, we can use the same scattering matrix to find the final state after the second beam splitter,

$$\begin{pmatrix} a_2^\dagger \\ a_3^\dagger \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{e^{i\theta}}{\sqrt{2}} a_0^\dagger \\ \frac{i}{\sqrt{2}} a_1^\dagger \end{pmatrix} \tag{1.22}$$

$$= \begin{pmatrix} \frac{(e^{i\theta}-1)}{2} a_0^\dagger \\ \frac{i(e^{i\theta}+1)}{2} a_1^\dagger \end{pmatrix} \tag{1.23}$$

The probability at D_1 is $\left|\frac{(e^{i\theta}-1)}{2}\right|^2 = \sin^2\frac{\theta}{2}$. The probability at D_2 is $\left|\frac{i(e^{i\theta}+1)}{2}\right|^2 = \cos^2\frac{\theta}{2}$. This is a more rigorous description of single-photon interference.

1.2 Two-Photon Hong-Ou-Mandel Effect

The Hong-Ou-Mandel effect is a simple but novel phenomenon of quantum optics. It is an example where quantum interference leads to a non-classical result. Consider a symmetric beam 50-50 beam splitter. Now, two single-photon states hit the beam splitter from different sides. As shown in Fig. 3, the initial state is $|1\rangle_0|1\rangle_1$. What will be the output state? Since the total input photon number is two, the output state could be $|2\rangle_2|0\rangle_3$, $|0\rangle_2|2\rangle_3$, and $|1\rangle_2|1\rangle_3$. Here, the subscripts of the ket denote on which side the photon state is, as in Fig. 3. Hence, a general expression for the output state is the superposition of the above possible states,

$$|\text{output}\rangle = \alpha|2\rangle_2|0\rangle_3 + \beta|0\rangle_2|2\rangle_3 + \gamma|1\rangle_2|1\rangle_3. \tag{1.24}$$

From the classical point of view, the state $|1\rangle_2|1\rangle_3$ seems the most possible state. However, in 1987, Hong et al. showed that for a symmetric beam splitter, $\gamma = 0$, that is, the probability of the state $|1\rangle_2|1\rangle_3$ is zero.[2] Specifically, the output state is

$$\begin{aligned}
|\text{output}\rangle &= U^T a_0^\dagger|0\rangle U^T a_1^\dagger|0\rangle &(1.25)\\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} a_0^\dagger \\ 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 \\ a_1^\dagger \end{pmatrix}|0\rangle &(1.26)\\
&= \frac{1}{2}\left(a_2^\dagger + ia_3^\dagger\right)\left(ia_2^\dagger + a_3^\dagger\right)|0\rangle &(1.27)\\
&= \frac{i}{2}\left[\left(a_2^\dagger\right)^2 + \left(a_3^\dagger\right)^2\right]|0\rangle &(1.28)\\
&= \frac{i}{\sqrt{2}}(|2\rangle_2|0\rangle_3 + |0\rangle_2|2\rangle_3) &(1.29)
\end{aligned}$$

Surprisingly, two identical single photons are bunched after hitting a beam splitter from different sides. In the linear optical quantum computing, the Hong-Ou-Mandel effect is the mechanism for the logic gates.

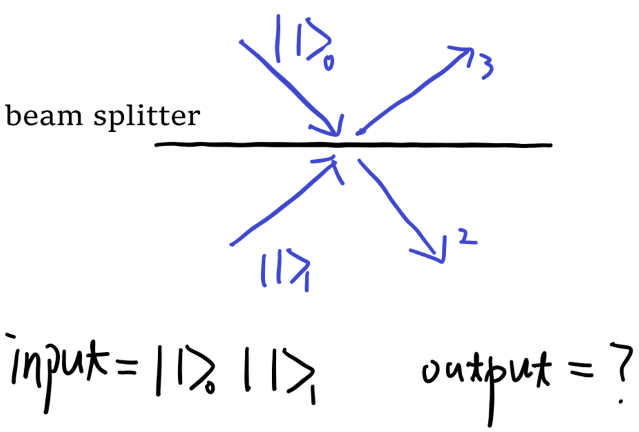

Figure 3: Hong-Ou-Mandel effect.

1.3 N-Photon

Let the initial state be $|0\rangle_0|N\rangle_1 = \frac{(a_1^\dagger)^N}{\sqrt{N!}}|0\rangle$. After a beam splitter, the state becomes

$$\frac{(Ua_1^\dagger)^N}{\sqrt{N!}}|0\rangle = \frac{(ta_2^\dagger + ra_3^\dagger)^N}{\sqrt{N!}}|0\rangle. \tag{1.30}$$

1.4 Coherent States

Let the initial state be $|0\rangle_0|\alpha\rangle_1 = D_1[\alpha]|0\rangle = e^{\alpha a_1^\dagger - \alpha^* a_1}|0\rangle$. After a beam splitter, the state becomes

$$\begin{aligned} e^{\alpha U a_1^\dagger - \alpha^* U^\dagger a_1}|0\rangle &= e^{\alpha t a_2^\dagger - \alpha^* t^* a_2} e^{\alpha r a_3^\dagger - \alpha^* r^* a_3}|0\rangle & (1.31)\\ &= D_2[t\alpha]D_3[r\alpha]|0\rangle & (1.32)\\ &= |t\alpha\rangle_2 |r\alpha\rangle_3. & (1.33) \end{aligned}$$

The input of a coherent state is split into a product of two coherent states. Unlike the single-photon case, this state is not entangled.

Consider a Mach–Zehnder interferometer with two 50:50 beam splitters of $r = \frac{i}{\sqrt{2}}$ and $t = \frac{1}{\sqrt{2}}$. Let the initial state be $|0\rangle_0|\alpha\rangle_1$. After the first beam splitter, the state becomes

$$\left|\frac{i\alpha}{\sqrt{2}}\right\rangle_2 \left|\frac{\alpha}{\sqrt{2}}\right\rangle_3. \tag{1.34}$$

When the state arrives at the second beam splitter, the state becomes

$$\left|\frac{i\alpha}{\sqrt{2}}\right\rangle_2 \left|\frac{e^{i\theta}\alpha}{\sqrt{2}}\right\rangle_3. \tag{1.35}$$

where θ is a phase shift due to the difference of the two paths. The final state after the second beam splitter is (see the figure: a_2 is the new a_1 and a_3 is the new a_0)

$$\left|\frac{\left(e^{i\theta} - 1\right)}{2}\right\rangle_2 \left|\frac{i\left(e^{i\theta} + 1\right)}{2}\right\rangle_3. \tag{1.36}$$

The intensity at D_1 is $|\alpha|^2 \left|\frac{e^{i\theta}-1}{2}\right|^2 = \sin^2\frac{\theta}{2}|\alpha|^2$. The intensity at D_2 is $|\alpha|^2 \left|\frac{e^{i\theta}+1}{2}\right|^2 = \cos^2\frac{\theta}{2}|\alpha|^2$. The two output beams are both coherent states. Thus, the phase θ can be obtained by

$$\frac{I_2 - I_1}{|\alpha|^2} = \cos\theta. \tag{1.37}$$

However, the amplitudes have uncertainty $\sigma(n) = \sqrt{\bar{n}}$. Thus, the phase has the uncertainty $\sigma(\theta) \sim \frac{1}{\sqrt{\bar{n}}}$. In experiments, we would like to use a coherent light (laser) with a well-defined phase and a strong intensity such that the uncertainty in the phase is small. However, a strong-intensity light may lead to more noise, such as radiation pressures, thermal noises, etc. To solve this dilemma, lights with small $\sigma(n)$ are used. These lights are non-classical lights.

References

[1] Z. Y. Ou and L. Mandel, Derivation of reciprocity relations for a beam splitter from energy balance, American Journal of Physics 57, 66 (1989)

[2] C. K. Hong; Z. Y. Ou & L. Mandel . "Measurement of subpicosecond time intervals between two photons by interference." Phys. Rev. Lett. 59 (18): 2044–2046 (1987)