Coherent State and Phase Space Descriptions

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Contents

Number (Fock) states are extremely quantum in the sense that there is no classical analog. For example, the expectation value of electric field is time-independent. Hence, pure number states can easily exhibit the quantum properties such as superposition and entanglement. The approach of using these discrete states for encoding qubit is called discrete variable (DV) encoding. However, there are two main disadvantages of using DV encoding. First, DV states may be too fragile. Second, it is hard to prepare such extremely quantum states. These lead to the development of using continuous variable (CV) Encoding.

In the example of a harmonic oscillator, n is the discrete number, and x, p are the continuous variables. All the quantum states can be represented in the number basis and in the continuous variable basis.

In this note, we will use the quadrature operators X and Y to describe quantum state. They can be regarded as continuous variables. Another reason to use CV is that CV has the classical counterpart so that it is easier to have intuitive pictures.

Let's explore some methods for encoding qubits using light.
Discrete Variable (DV) Encoding: In DV encoding, polarization or photon-number degrees of freedom are employed to create and encode qubits from light. Examples include using horizontal and vertical polarizations of a single photon or the vacuum and single-photon excitation of a quantized light mode. However, DV encoding can be expensive for multiple-qubit operations, such as Bell-state measurements in optical quantum computation and communication.

Continuous Variable (CV) Encoding: CV encoding views the structure of a quantized field of light in a continuous-variable space. Qubit encoding using coherent states of opposite phases (such as $|\pm\alpha\rangle$ or their superpositions like $|\alpha\rangle \pm |-\alpha\rangle$) has been investigated. These superposed coherent states are known as cat-state qubits, inspired by Schrödinger's cat paradox. Cat-state qubits allow nearly deterministic Bell-state measurements and two-qubit gate operations in a simpler way, but they are sensitive to detection inefficiency and decoherence due to photon loss.

Hybrid Approaches: Combining DV and CV approaches provides powerful tools for photonic quantum computing and communication. Previous work demonstrated DV-CV hybrid entanglement and transformation between DV and CV states. Recent research reports a qubit converter that transforms a DV qubit to a cat-state qubit, bridging the gap between discrete and continuous-variable representations.

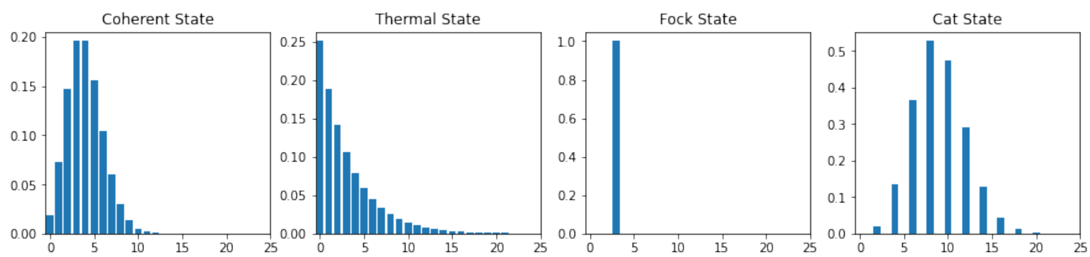

Figure 1: Photon counting of different quantum lights.

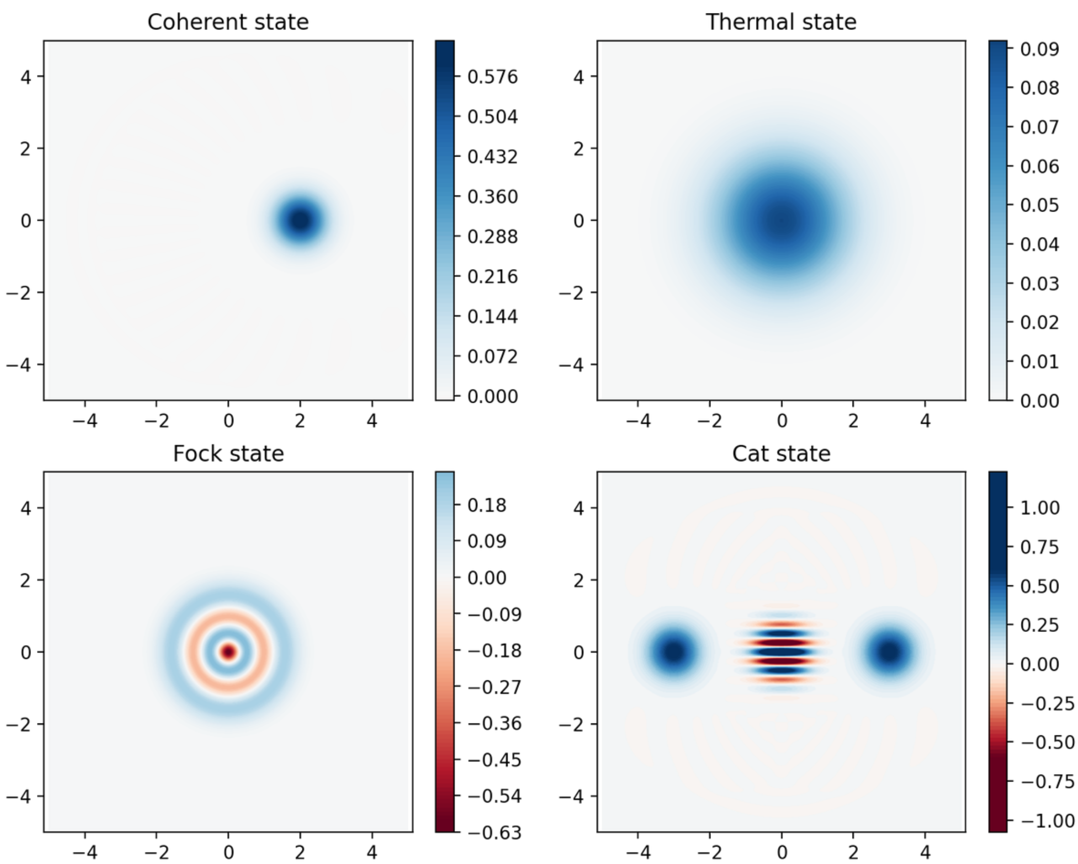

Figure 2: Phase space of different quantum lights.

1 Phase Space Pictures

The state of a classical particle is fully determined by its x and p. A useful way to represent the states is the phase space (x,p), where the horizontal axis is x and the vertical axis is p. A state of a classical particle is one point in the phase phase. The time evolution of a state is the trajectory in the phase space. The trajectory $(x(t),p(t))$ contains all the information of the particle. The classic example is the harmonic oscillator with

$$x(t) = x_0 \cos(\omega t + \phi), \tag{1.1}$$
$$p(t) = -\omega x_0 \sin(\omega t + \phi), \tag{1.2}$$

or in the dimensionless expression

$$\tilde{x}(t) = \frac{x(t)}{x_0} = \cos(\omega t + \phi), \tag{1.3}$$
$$\tilde{p}(t) = \frac{p(t)}{\omega x_0} = -\sin(\omega t + \phi). \tag{1.4}$$

The state travels along the trajectory, a unit circle (see Fig. 3).

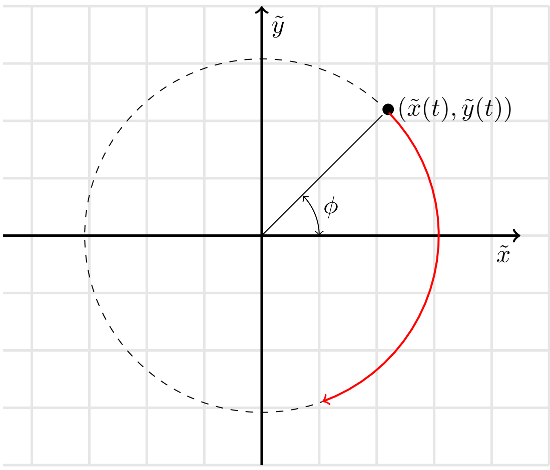

Figure 3: A classical state is a point in the phase space. The motion of a state is a trajectory. In the case of a harmonic oscillator, the trajectory is a circle.

An ensemble of classical particles is described by the phase space probability density function $f(x,p)$, where to find a particle with a position x and a momentum p is given by

$$f(x,p)dxdp, \tag{1.5}$$

and the normalization condition is

$$\int dx \int dp f(x,p)dxdp = 1. \tag{1.6}$$

Classically, the function $f(x,p)$ of a pure state, i.e., a single particle, is a delta function $f(x,p) = \delta(x-x_0)\delta(p-p_0)$. We have made the analogies $x \leftrightarrow X$ and $y \leftrightarrow Y$. One question arises: can we define a function similar to $f(x,p)$ to describe states or ensembles of photons? The problem is that a quantum state can not have exact X and Y simultaneously. Thus, a quantum state is not a single point in the phase space. Recall the relations

$$X = \frac{a+a^\dagger}{2}, \tag{1.7}$$

$$Y = \frac{a-a^\dagger}{2i}. \tag{1.8}$$

For a coherent state $|\alpha\rangle$, we have the relations

$$\langle X\rangle = \frac{\alpha+\alpha^*}{2}, \tag{1.9}$$

$$\langle Y\rangle = \frac{\alpha-\alpha^*}{2i}. \tag{1.10}$$

As you can show $\sigma(X) = \sigma(Y) = 1/2$ for a coherent state, which means that a state in the phase space is not a point but a blurred circular cloud (see Fig. 4). The size of the cloud reflects the uncertainty relations. Coherent states are the states satisfying the minimum uncertainty relations. Generally, an arbitrary state can have an extensive distribution in the phase space.

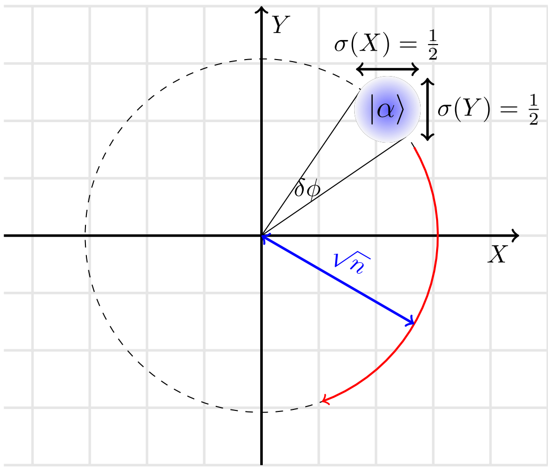

Figure 4: A coherent state is a fuzzy circle in the phase space.

A mapping of a state $|\psi\rangle$ or an ensemble to a distribution in the phase space (X, Y) (or equivalently the complex α space.) provides a physical picture. However, a mapping $|\psi\rangle \to f(X, Y)$ is not uniquely defined. The problem is that X and Y are non-commutative operators. Many attempts exist to define a probability density $f(X, Y)$ or $f(\alpha)$. We are going to introduce the three most used definitions,

- Wigner distribution
- Q-function
- P-function

Note that the definitions and calculations of these functions are quite mathematically involved. These functions serve as quantitative tools to describe the phase space probability densities. It is fine to have a qualitative picture in mind first and know more calculations when it is needed.

2 Coherent States

We have shown that the number states $|n\rangle$ do not behave similarly to the classical fields. For example, the expectation value $\langle n|\hat{\mathbf{E}}|n\rangle$ is not only static but also zero. A classical field is a field whose amplitude is a harmonic function of t, i.e., $\exp(\pm i\omega t)$. Since the number states form a complete set of the basis vectors, all the photon states, including the classical field, can be written on a number-state basis. Hence, we write a classical field $\mathbf{E}_{\text{cl}}(\mathbf{r}, t)$ as a superposition of the number states,

$$|\text{classical}\rangle = \sum_n C_n |n(t)\rangle = \sum_n C_n e^{-in\omega t} |n(0)\rangle. \tag{2.1}$$

The coefficients C_n are to be determined to satisfy the following properties. The classical field $\mathbf{E}_{\text{cl}}(\mathbf{r}, t)$ is the expectation value of the electric field of the classical state,

$$\mathbf{E}_{\rm cl}(\mathbf{r},t) = \langle \text{classical}|\hat{\mathbf{E}}|\text{classical}\rangle, \tag{2.2}$$

where for a mode of frequency ω, the classical field $\mathbf{E}_{\rm cl}(\mathbf{r},t)$ is sinusoidal,

$$\mathbf{E}_{\rm cl}(\mathbf{r},t) = \boldsymbol{\mathcal{E}}_\omega(\mathbf{r})e^{-i\omega t+\phi}. \tag{2.3}$$

A classical field has two features: the harmonic oscillation term $e^{-i\omega t}$ and the phase ϕ. Although the expectation value by Eq. (2.3) defines the exact values of the amplitude and the phase, the amplitude and phase of the electric field of a state $|\psi\rangle$, in general, have uncertainties. Hence, the amplitude and phase of a state should be described by probability distributions.

Note 1: Coherent State

A coherent state is a most classical state in which the amplitude is a finite constant, the phase grows as ωt, and the uncertainties of the amplitude and phase are minimized.

Below, we first discuss how to obtain the phase distribution of a state $|\psi\rangle$, and find the coefficient C_n of a coherent state.

2.1 Quantum Phase

In quantum optics, the electric field $\mathbf{E}$ of an arbitrary photon state $|\psi\rangle$ has the uncertainties in both its amplitude and phase, that is, $\langle E^2\rangle \neq 0$ and $\langle \phi^2\rangle \neq 0$. Indeed, we have not talked about obtaining ϕ of a photon state $|\psi\rangle$. Note that the phase ϕ is not the phase of a wavefunction but the phase of the electric field. Since $\mathbf{E}$ is an operator but not a number, it turns out that there are many definitions of the phase ϕ. Moreover, the phase ϕ of a state $|\psi\rangle$ is not a single value but a distribution with a finite variance. We will define a phase distribution $\mathcal{P}(\phi)$ where $\mathcal{P}(\phi)d\phi$ is the probability of finding the state to have a phase ϕ. Here, we follow Susskind and Glogower's approach to obtain the phase distribution. The Susskind-Glogower operators are defined by

$$A \equiv (aa^\dagger)^{-\frac{1}{2}}a = (N+1)^{-\frac{1}{2}}a, \tag{2.4}$$

$$A^\dagger \equiv a^\dagger(aa^\dagger)^{-\frac{1}{2}} = a^\dagger(N+1)^{-\frac{1}{2}}. \tag{2.5}$$

If we temporarily treat a as a complex number, $a = |a|\exp i\phi$, the operator A will look as $A = \exp i\phi$. This is the motivation of the definitions, which is to make the operator A taking out the phase factor $\exp i\phi$ of a state. The properties of the SG operators are

$$A|n\rangle = \begin{cases} |n-1\rangle, & n \neq 0, \\ 0, & n = 0, \end{cases} \tag{2.6}$$

$$A^\dagger|n\rangle = |n+1\rangle, \tag{2.7}$$

in the number state bases,

$$A = \sum_n |n\rangle\langle n+1|, \tag{2.8}$$

$$A^\dagger = \sum_n |n+1\rangle\langle n|, \tag{2.9}$$

$$AA^\dagger = 1, \tag{2.10}$$

$$A^\dagger A = 1 - |0\rangle\langle 0|. \tag{2.11}$$

The eigenstate of A is $|\phi\rangle$,

$$A|\phi\rangle = e^{i\phi}|\phi\rangle. \tag{2.12}$$

The state $|\phi\rangle$ in the number states is

$$|\phi\rangle = \sum_n e^{in\phi}|n\rangle. \tag{2.13}$$

The state given by Eq. (2.13) is not normalized. The states $|\phi\rangle$ and $|\phi'\rangle$ are not orthogonal, that is, $\langle\phi'|\phi\rangle \neq 0$. Using the fact

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(n-n')\phi}\, d\phi = \delta_{n,n'}, \tag{2.14}$$

we can show that

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi |\phi\rangle\langle\phi| = \mathbb{1}. \tag{2.15}$$

Derivation 1: Identity with Phase States

Let $|\psi\rangle$ be an arbitrary state. In the number state bases, it is

$$|\psi\rangle = \sum_n C_n|n\rangle. \tag{2.16}$$

Applying the operator in Eq. (2.15) on the state, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi|\phi\rangle\langle\phi|\psi\rangle = \sum_{C_n}\frac{1}{2\pi}\int_0^{2\pi} d\phi|\phi\rangle\langle\phi|n\rangle \tag{2.17}$$

$$= \frac{1}{2\pi}\sum_n\int d\phi|\phi\rangle C_n e^{-in\phi} \tag{2.18}$$

$$= \frac{1}{2\pi}\sum_{n,m}\int d\phi e^{im\phi}|m\rangle C_n e^{-in\phi} \tag{2.19}$$

$$= \sum_{n,m}\delta_{mn}C_n|m\rangle \tag{2.20}$$

$$= \sum_n C_n|n\rangle \tag{2.21}$$

$$= |\psi\rangle, \tag{2.22}$$

which proves the operator in Eq. (2.15) is an identity.

The phase distribution $\mathcal{P}(\phi)$ of a state $|\psi\rangle$ is

$$\mathcal{P}(\phi) \equiv \frac{1}{2\pi}|\langle\phi|\psi\rangle|^2 \tag{2.23}$$

$$= \frac{1}{2\pi}\left|\sum_n C_n e^{-in\phi}\right|^2. \tag{2.24}$$

The phase distribution $\mathcal{P}(\phi)$ is normalized,

$$\int_0^{2\pi}\mathcal{P}(\phi)d\phi = 1. \tag{2.25}$$

The phase distribution $\mathcal{P}(\phi)$ of an ensemble is

$$\mathcal{P}(\phi) = \frac{1}{2\pi}\langle\phi|\rho|\phi\rangle. \tag{2.26}$$

Note 2: Phase of a Phase State

The phase distribution function $\mathcal{P}(\phi)$ reveals the phase distribution of a state $|\psi\rangle$. Since N and A do not commute ($[N,A] = -A$), a state can not have a single phase but a phase distribution. The phase state $|\phi'\rangle$ is supposed to have a specific phase ϕ'. However, since the phase state is not normalized, It is not a physical tool but a mathematical one. We consider an approximate phase state, which is normalized,

$$|\phi'\rangle_{\text{app}} \equiv \sum_{n=0}^{N_{\text{max}}}\frac{e^{in\phi'}|n\rangle}{\sqrt{N_{\text{max}}+1}}. \tag{2.27}$$

The phase distribution function of $|\phi'\rangle_{\rm app}$ is

$$\mathcal{P}(\phi) = \frac{1}{2(N_{\rm max}+1)\pi}\left|\frac{\sin\left[\frac{(N_{\rm max}+1)(\phi-\phi')}{2}\right]}{\sin\left[\frac{\phi-\phi'}{2}\right]}\right|^2. \tag{2.28}$$

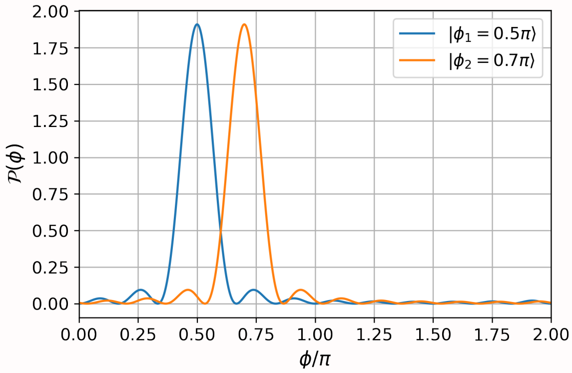

Figure 5: Phase distribution functions of $|\phi_1 = 0.5\pi\rangle$ and $|\phi_2 = 0.7\pi\rangle$. The maximum number is $N_{\rm max} = 12$.

```python
import matplotlib
import matplotlib.pyplot as plt
import numpy as np
# Data for plotting
phi1 = 0.5 * np.pi
phi2 = 0.7 * np.pi
Nmax = 12
phi = np.arange(0.0, 2.0 * np.pi, 0.01)
# define the phase distribution function
def phase_dist_func(x,y):
    return np.sin(Nmax*(x-y)/2)**2./np.sin((x-y)/2)**2/Nmax/(2*np.pi)
phase_dist_1 = phase_dist_func(phi,phi1)
phase_dist_2 = phase_dist_func(phi,phi2)
## plot
fig, ax = plt.subplots()
ax.plot(phi, phase_dist_1,label=r'$|\phi_1=0.5\pi\rangle$')
## r: raw string
ax.plot(phi, phase_dist_2,label=r'$|\phi_2=0.7\pi\rangle$')
## r: raw string
ax.set(xlabel='$\phi$ ', ylabel='$\mathcal{P}(\phi)$',
       title='Phase Distribution Function of a Phase State')
ax.grid()
plt.legend()
fig.savefig("phase_dist.png", dpi=300)
plt.show()
```

Figure 6: Python codes.

Exercise 1: Phase Distribution Function

Show Eq. (2.28). Use Eq. (2.24). The summation is a geometric series.

2.2 Coherent States

We have shown that a phase state $|\phi\rangle$ has a well-defined phase. However, as a classical field, not only the phase but also the field amplitude should be well-defined; that is, we expect that $\langle \mathbf{E}\rangle$ does not vanish, and $\sigma(\mathbf{E})$ is small. Since the phase states are neither normalized nor physical, we must find other states.

The goal is to find the states $|\alpha\rangle$ such that the expectation of the electric field $\langle\alpha|\mathbf{E}|\alpha\rangle$ is proportional to the classical field $\mathcal{E}_\omega(\mathbf{r}) + \mathcal{E}^*_\omega(\mathbf{r})$. By observing that

$$\mathbf{E}_\omega(\mathbf{r}) = \frac{\left[\mathcal{E}_\omega(\mathbf{r})a + \mathcal{E}^*_\omega(\mathbf{r})a^\dagger\right]}{2}, \tag{2.29}$$

one finds that if the states $|\alpha\rangle$ are the eigenstates of the annihilation operator a,

$$a|\alpha\rangle = \alpha|\alpha\rangle, \tag{2.30}$$

with the eigenvalues α, the expectation value $\langle\alpha|\mathbf{E}|\alpha\rangle$ is the same as the classical field. Since the operator a is not hermitian, the eigenvalues α can generally be complex numbers. It turns out that the states $|\alpha\rangle$, called "coherent states", are the most classical states. Let's find out the coherent states in the number state bases. We expand the coherent states as

$$|\alpha\rangle = \sum_n C_n|n\rangle, \tag{2.31}$$

and plug it in Eq. (2.30),

$$a|\alpha\rangle = \sum_n C_n a|n\rangle = \alpha\sum_n C_n|n\rangle \tag{2.32}$$

$$\Rightarrow \sum_n C_n\sqrt{n}|n-1\rangle = \alpha\sum_n C_n|n\rangle. \tag{2.33}$$

We obtain

$$C_{n+1} = \alpha\frac{C_n}{\sqrt{n+1}}, \tag{2.34}$$

$$C_n = \frac{\alpha^n}{\sqrt{n!}}C_0, \tag{2.35}$$

and thus

$$|\alpha\rangle = C_0\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{2.36}$$

The normalization condition fixes the coefficient C_0,

$$\langle\alpha|\alpha\rangle = |C_0|^2\sum_{m,n}\frac{\alpha^n(\alpha^*)^m}{\sqrt{n!m!}}\langle m|n\rangle, \tag{2.37}$$

where one finds

$$C_0 = e^{-\frac{|\alpha|^2}{2}}. \tag{2.38}$$

The coherent states are

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_n\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{2.39}$$

$$= e^{-\frac{|\alpha|^2}{2}}\sum_n\frac{\alpha^n(a^\dagger)^n}{n!}|0\rangle \tag{2.40}$$

$$= e^{-\frac{|\alpha|^2}{2}}e^{\alpha a^\dagger}|0\rangle. \tag{2.41}$$

Exercise 2: Normalization Constant

Show Eq. (2.38). Begin with Eq. (2.37).

The expectations are

$$\langle\alpha|\mathbf{E}|\alpha\rangle = \left\langle\alpha\left|\frac{\left[\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})a + \boldsymbol{\mathcal{E}}_\omega^*(\mathbf{r})a^\dagger\right]}{2}\right|\alpha\right\rangle \tag{2.42}$$

$$= \mathrm{Re}[\alpha\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})] \tag{2.43}$$

$$\langle\alpha|\mathrm{abs}[\mathbf{E}]^2|\alpha\rangle = \left\langle\alpha\left|\frac{\mathrm{abs}\left[\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})a + \boldsymbol{\mathcal{E}}_\omega^*(\mathbf{r})a^\dagger\right]^2}{4}\right|\alpha\right\rangle$$

$$= \mathrm{abs}[\mathrm{Re}[\alpha\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})]]^2 + \frac{|\mathcal{E}_\omega(\mathbf{r})|^2}{4}. \tag{2.44}$$

The standard deviation of the electric field is

$$\sigma(\mathbf{E}) = \frac{|\mathcal{E}_\omega(\mathbf{r})|}{2}. \tag{2.45}$$

The standard deviation is relatively small compared to the field amplitude when $|\alpha|$ is large. We can see this by dividing $\sigma(\mathbf{E})$ with $\langle\alpha|\mathbf{E}|\alpha\rangle$,

$$\frac{\sigma(\mathbf{E})}{\langle\alpha|\mathbf{E}|\alpha\rangle} = \frac{|\mathcal{E}_\omega(\mathbf{r})|}{2\mathrm{Re}[\alpha\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})]}. \tag{2.46}$$

The coherent states $|\alpha\rangle$ indeed have the minimum uncertainty. Using the quadrature operators X and Y, one can show that the coherent states have

$$\sigma(X) = \sigma(Y) = \frac{1}{2}. \tag{2.47}$$

Exercise 3: Uncertainty Relations

Show Eq. (2.47). Hints:

(a) $\langle\alpha|X|\alpha\rangle = \frac{\alpha+\alpha^*}{2}$

(b) $\langle\alpha|X^2|\alpha\rangle = \left(\frac{\alpha+\alpha^*}{2}\right)^2 + \frac{1}{4}$. Note that $(a+a^\dagger)^2 = a^2 + 2a^\dagger a + \left(a^\dagger\right)^2 + 1$

The physical meaning of α is the dimensionless amplitude, which is seen from that the average number $\bar{n}$ of a coherent state $|\alpha\rangle$ is

$$\bar{n} = \langle\alpha|N|\alpha\rangle = \langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2. \tag{2.48}$$

The standard deviation $\sigma(N)$ is

$$\sigma(N) = |\alpha| = \bar{n}^{\frac{1}{2}}. \tag{2.49}$$

The standard deviation $\sigma(N)$ over the average number $\bar{n}$ is

$$\frac{\sigma(N)}{\bar{n}} = \bar{n}^{\frac{-1}{2}}. \tag{2.50}$$

The probability p_n of measuring the number state $|n\rangle$ is a Poisson distribution

$$p_n = |C_n|^2 = e^{-|\alpha^2|}\frac{|\alpha|^{2n}}{n!} = e^{-\bar{n}}\frac{\bar{n}^n}{n!}. \tag{2.51}$$

The phase distribution function $\mathcal{P}(\phi)$ of a coherent state is

$$\mathcal{P}(\phi) = \frac{e^{-|\alpha|^2}}{2\pi}\left|\sum_n \frac{\alpha^n}{\sqrt{n!}}\right|^2. \tag{2.52}$$

Let $\alpha = |\alpha|^{i\bar{\phi}}$. One can show that as $\bar{n} = |\alpha|^2$ is large, the distributions become approximately the Gaussian distributions (See Ref. [1]),

$$p_n \simeq (2\pi\bar{n})^{-1/2}\, e^{-\frac{(n-\bar{n})^2}{2\bar{n}}}, \tag{2.53}$$

$$\mathcal{P}(\phi) \simeq \sqrt{\frac{2\bar{n}}{\pi}}\, e^{-2\bar{n}(\phi-\bar{\phi})^2}. \tag{2.54}$$

2.3 Displaced Vacuum States

The physical meaning of α is the dimensionless (complex) amplitude of a coherent state. The vacuum state is indeed a coherent state in the limit $\alpha \to 0$. Conversely, a coherent state is obtained by changing the complex amplitude α of the vacuum state. Mathematically, such a shift of α is done by the displacement operator $D(\alpha)$,

$$|\alpha\rangle = D(\alpha)|0\rangle. \tag{2.55}$$

The displacement operator $D(\alpha)$ has the explicit form

$$D(\alpha) = \exp\Big(\alpha a^\dagger - \alpha^* a\Big). \tag{2.56}$$

To show this, first consider the special case of Baker–Campbell–Hausdorff formula, if

$$[A,[A,B]] = [B,[A,B]] = 0, \tag{2.57}$$

we have

$$e^{A+B} = e^{-\frac{1}{2}[A,B]}e^A e^B \tag{2.58}$$

$$= e^{\frac{1}{2}[B,A]}e^B e^A. \tag{2.59}$$

With $A = \alpha a^\dagger$, $B = -\alpha^* a$, and $[A,B] = |\alpha|^2$, the displacement operator $D(\alpha)$ becomes

$$D(\alpha) = e^{-\frac{|\alpha|^2}{2}}e^{\alpha a^\dagger}e^{-\alpha^* a}. \tag{2.60}$$

Using the relations

$$e^{-\alpha^* a}|0\rangle = \left(\mathbb{1} - \alpha^* a + \frac{(-\alpha^* a)^2}{2!} + \ldots\right)|0\rangle = |0\rangle, \tag{2.61}$$

we obtain

$$\begin{aligned} D(\alpha)|0\rangle &= e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} e^{-\alpha^* a}|0\rangle && (2.62)\\ &= e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger}|0\rangle && (2.63)\\ &= e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger}|0\rangle && (2.64)\\ &= e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n (a^\dagger)^n}{n!}|0\rangle && (2.65)\\ &= e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle && (2.66)\\ &= |\alpha\rangle. && (2.67) \end{aligned}$$

The displacement operator $D(\alpha)$ is unitary and satisfies the relation

$$\begin{aligned} D(\alpha)D^\dagger(\alpha) = D^\dagger(\alpha)D(\alpha) = \mathbb{1}, && (2.68)\\ D^\dagger(\alpha) = D(-\alpha). && (2.69) \end{aligned}$$

The displacement operators satisfy the law of addition; operations by two subsequent displacement operator $D(\alpha)$ and $D(\beta)$ give a total displacement operator

$$D(\alpha)D(\beta) = e^{i\mathrm{Im}[\alpha\beta^*]} D(\alpha + \beta). \tag{2.70}$$

We see that the total displacement is $\alpha + \beta$, that is, the sum of the displacements of the individual displacement operators. An extra phase $\mathrm{Im}[\alpha\beta^*]$ is the quantum feature, and note although the total displacement does not depend on the order of the operators, the phase does depend.

Note 3: Displacement Operator

For now, a displacement operator is just a mathematical tool. Later, as we learn light-matter interaction, we will know that a displacement operator is the evolution operator of a sinusoidal driving source, $\mathcal{H}_i(t) \sim \sin(\omega t + \phi)$. If we turn on a sinusoidal driving source, the vacuum state will be shifted in the complex α space. This is one method to generate coherent states.

2.4 Dynamics of Coherent States

The dynamics of a coherent state $|\alpha\rangle$ is given by the Schrödinger's picture,

$$\begin{aligned} |\alpha(t)\rangle &= e^{-i\frac{\mathcal{H}t}{\hbar}}|\alpha(0)\rangle && (2.71)\\ &= e^{-i\frac{\omega}{2}t} e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n e^{-in\omega t}}{\sqrt{n!}}|n\rangle && (2.72)\\ &= e^{-i\frac{\omega}{2}t}|\alpha(0)e^{-i\omega t}\rangle. && (2.73) \end{aligned}$$

Thus, the amplitude $\alpha(t)$ is

$$\alpha(t) = \alpha(0)e^{-i\omega t}. \tag{2.74}$$

Although every photon mode $\mathcal{E}_\omega(\mathbf{r})$ can be pretty different from one system to another system, we can use the dimensionless quadrature operators $\hat{X}$ and $\hat{Y}$ to describe the dynamics. Recall that $\hat{X}$ is analogous to the position operator, and $\hat{Y}$ is analogous to the momentum operator. We can express a coherent state on the X basis,

$$\psi_\alpha(X) = \langle X|\alpha\rangle, \tag{2.75}$$

where $|X\rangle$ is the eigenvector of X

$$\hat{X}|X\rangle = X|X\rangle. \tag{2.76}$$

To find $\psi_\alpha(X)$, we begin with

$$\langle X|a|\alpha\rangle = \alpha\langle X|\alpha\rangle \tag{2.77}$$
$$\Rightarrow \langle X|\hat{X} + i\hat{Y}|\alpha\rangle = \alpha\langle X|\alpha\rangle \tag{2.78}$$
$$\Rightarrow \left(X + \frac{\partial}{\partial X}\right)\langle X|\alpha\rangle = \alpha\langle X|\alpha\rangle \tag{2.79}$$
$$\Rightarrow \frac{\partial \psi_\alpha(X)}{\partial X} = (\alpha - X)\psi_\alpha(X) \tag{2.80}$$
$$\Rightarrow \psi_\alpha(X) = \sqrt{\frac{2}{\pi}}e^{-\frac{(X-\mathrm{Re}[\alpha])^2}{2}}e^{i\mathrm{Im}[\alpha]X}, \tag{2.81}$$

We used the normalization condition to derive the last step. The wavefunction $\psi_\alpha(X)$ is a Gaussian distribution, and its peak position is

$$X_p(t) = \mathrm{Re}[\alpha(t)] \tag{2.82}$$
$$= |\alpha(0)|\cos(\phi_0 - \omega t). \tag{2.83}$$

with $\alpha(0) = |\alpha(0)|e^{i\phi_0}$. The peak position $X_p(t)$ is the same as that of a classical harmonic oscillator. Also the wavefunction $\psi_\alpha(X)$ has an minimum spreads of X and P. Thus, a coherent state is the most classical state.

Summary 1: Coherent States

Coherent states are

- eigenstates of the annihilation operator a.
- displaced vacuum states.
- most classical states whose phase and amplitude distributions are narrow.
- most classical states whose X and Y distributions are narrow.

- minimum uncertainty states.

2.5 Properties of Coherent States

2.5.1 Orthogonality

Two coherent states $|\alpha\rangle$ and $|\beta\rangle$ are not orthogonal,

$$\langle\beta|\alpha\rangle = e^{-\frac{(|\alpha|^2+|\beta|^2)}{2}}\sum_{n,m}\frac{(\beta^*)^m(\alpha)^n}{\sqrt{m!n!}}\langle m|n\rangle \tag{2.84}$$

$$= e^{-\frac{(|\alpha|^2+|\beta|^2)}{2}}\sum_{n}\frac{(\beta^*)^n(\alpha)^n}{n!} \tag{2.85}$$

$$= e^{-\frac{(|\alpha|^2+|\beta|^2)}{2}}e^{\beta^*\alpha} \tag{2.86}$$

$$= e^{-\frac{|\alpha-\beta|^2}{2}}e^{\frac{\beta^*\alpha-\beta\alpha^*}{2}}, \tag{2.87}$$

which does not vanish.

2.5.2 Identity

The identity can be expressed with the coherent states,

$$\int\frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha| \equiv \int\frac{d\mathrm{Re}[\alpha]d\mathrm{Im}[\alpha]}{\pi}|\alpha\rangle\langle\alpha| = \mathbb{1}. \tag{2.88}$$

Derivation 2: Identity with Coherent States

The proof of Eq. (2.88) is as follows. Let $\alpha = re^{\phi}$ and.

$$d\alpha^2 = d\mathrm{Re}[\alpha]d\mathrm{Im}[\alpha] = rdrd\phi. \tag{2.89}$$

The left hand side of Eq. (2.88) becomes

$$\int\frac{rdrd\phi}{\pi}|\alpha\rangle\langle\alpha| = \int\frac{rdrd\phi}{\pi}e^{-r^2}\sum_{m,n}\frac{e^{i(n-m)\phi}r^{m+n}}{n!}|m\rangle\langle n| \tag{2.90}$$

$$= \sum_n\frac{\int dre^{-r^2}2r^{2n+1}}{n!}|n\rangle\langle n| \tag{2.91}$$

$$= \sum_n\frac{\int due^{-u}u^n}{n!}|n\rangle\langle n| \tag{2.92}$$

$$= \sum_n|n\rangle\langle n| = \mathbb{1}. \tag{2.93}$$

2.5.3 Coherent State Representations of Operators

Any operator X can be expressed in the coherent state bases with the identity Eq. (2.88),

$$X = \int \frac{d^2\alpha}{\pi} \int \frac{d^2\beta}{\pi} |\alpha\rangle\langle\alpha|X|\beta\rangle\langle\beta|. \tag{2.94}$$

However, coherent states are not orthogonal, so the coherent states form an **over-complete** set of bases.[1] It is possible to write X in the coherent state diagonal form.

An operator X is uniquely determined by $\langle\alpha|X|\alpha\rangle$. The diagonal element $\langle\alpha|X|\alpha\rangle$ in the number state basis is

$$\langle\alpha|X|\alpha\rangle = \exp -|\alpha|^2 \sum_{m,n} \frac{\langle n|X|m\rangle\alpha^m(\alpha^*)^n}{\sqrt{m!n!}}, \tag{2.95}$$

indicating that $\langle\alpha|X|\alpha\rangle$ contains all the information of the elements $\langle n|X|m\rangle$, which forms a complete set.

Coherent state diagonal representation. Suppose that X has a series expansion of a and $a^\dagger$ in the antinormal ordering,

$$X = \sum_{mn} \chi^A_{nm} a^n (a^\dagger)^m, \tag{2.96}$$

where χ^A_{nm} is a c-number. The superscript A denotes the antinormal ordering. Inserting the identity, we obtain

$$X = \sum_{mn} \chi^A_{nm} a^n \left(\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| \right) (a^\dagger)^m \tag{2.97}$$

$$= \int d^2\alpha \chi^A(\alpha) |\alpha\rangle\langle\alpha| \tag{2.98}$$

where

$$\chi^A(\alpha) = \frac{1}{\pi} \sum_{mn} \chi^A_{nm} \alpha^n (\alpha^*)^m, \tag{2.99}$$

is a c-number.

3 Phase Space Distributions

Given a density matrix ρ, there are three important distribution functions which are the quantum analogs of the classical probability density $f(x,p)$.

[1]See Se. 5.4. of Ref. [2] for a more rigorous discussion.

3.1 Wigner Distribution

The Wigner function $W(\alpha)$ is defined as

$$W(\alpha) = \int \frac{d^2\eta}{\pi^2} e^{\eta^*\alpha - \eta\alpha^*} \chi_W(\eta), \tag{3.1}$$

where the characteristic function $\chi_W(\eta)$ is

$$\chi_W(\eta) = \mathrm{Tr}\left[\rho e^{\eta a^\dagger - \eta^* a}\right]. \tag{3.2}$$

Exercise 4: Normalization

Show that

$$\int \frac{d^2\alpha}{\pi^2} e^{\eta^*\alpha - \eta\alpha^*} = \delta_2(\eta) \equiv \delta(\mathrm{Re}[\eta])\delta(\mathrm{Im}[\eta]), \tag{3.3}$$

and use the result and Eq. (3.1) to show

$$\int d^2\alpha W(\alpha) = 1. \tag{3.4}$$

Hint: a delta function can be expressed as

$$\delta(x) = \frac{1}{2\pi} \int e^{ikx} dk. \tag{3.5}$$

Hint: let $\alpha = x + iy$ and use the identity $\delta(x) = \frac{1}{2\pi}\int e^{iqx} dq$.

The ensemble average of an operator X in this representation is

$$\langle X \rangle = \int d^2\alpha \chi^W(\alpha) W(\alpha), \tag{3.6}$$

where

$$\chi^W(\alpha) = \sum_{n,m} \chi^W_{nm} \alpha^n (\alpha^*)^m \tag{3.7}$$

The coefficient χ^W_{nm} is the Weyl(symmetric)-ordering representation of an operator X,

$$X = \sum_{m,n} \chi^W_{nm} \left(\frac{(a^\dagger)^n a^m + a^m (a^\dagger)^n}{2} \right). \tag{3.8}$$

3.2 Glauber–Sudarshan P-function

The P-function is defined by

$$\rho = \int d^2\alpha P(\alpha) |\alpha\rangle\langle\alpha|, \tag{3.9}$$

and satisfies the normalization condition

$$1 = \mathrm{Tr}[\rho] = \int d^2\alpha P(\alpha). \tag{3.10}$$

The normal-ordering characteristic function can obtain the P-function

$$P(\alpha) = \int \frac{d^2\eta}{\pi^2} e^{\eta^*\alpha - \eta\alpha^*} \chi_N(\eta), \tag{3.11}$$

where the characteristic function $\chi_N(\eta)$ is

$$\chi_N(\eta) = \mathrm{Tr}\left[\rho e^{\eta a^\dagger} e^{-\eta^* a}\right]. \tag{3.12}$$

The ensemble average of an operator X in this representation is

$$\langle X \rangle = \int d^2\alpha \chi^N(\alpha) P(\alpha), \tag{3.13}$$

where

$$\chi^N(\alpha) = \sum_{n,m} \chi^N_{nm} (\alpha^*)^n \alpha^m \tag{3.14}$$

The coefficient χ^N_{nm} is the normal-ordering representation of an operator X,

$$X = \sum_{m,n} \chi^N_{nm} (a^\dagger)^n a^m. \tag{3.15}$$

Note 4: Classical and Nonclassical States

A state with $P(\alpha) < 0$ is a nonclassical state.

3.3 Q-function

The Q-function is defined by

$$Q(\alpha) = \frac{1}{\pi} \langle \alpha | \rho | \alpha \rangle, \tag{3.16}$$

which is always positive since it is the diagonal element of the density matrix. The function $Q(\alpha)$ satisfies

$$0 \leq Q(\alpha) \leq \frac{1}{\pi}, \tag{3.17}$$

and

$$\mathrm{Tr}[\rho] = \int d^2\alpha Q(\alpha) = 1. \tag{3.18}$$

	$W(\alpha)$	$Q(\alpha)$	$P(\alpha)$
coherent state $\|\alpha_0\rangle$	$\frac{2}{\pi}e^{-2\|\alpha-\alpha_0\|^2}$	$\frac{1}{\pi}e^{-\|\alpha-\alpha_0\|^2}$	$\delta^2(\alpha-\alpha_0)$
thermal ensemble	$\frac{1}{\pi(\bar{n}+1/2)}\exp\left(-\frac{\|\alpha\|^2}{\bar{n}+1/2}\right)$	$\frac{1}{\pi(\bar{n}+1)}\exp\left(-\frac{\|\alpha\|^2}{\bar{n}+1}\right)$	$\frac{1}{\pi(\bar{n})}\exp\left(-\frac{\|\alpha\|^2}{\bar{n}}\right)$
pure ensemble $\|1\rangle\langle 1\|$	$-(1-4\|\alpha\|^2)\frac{2}{\pi}e^{-2\|\alpha\|^2}$	$\frac{\|\alpha\|^2}{\pi}e^{-\|\alpha\|^2}$	singular

Table 1: Examples of $W(\alpha)$, $Q(\alpha)$, and $P(\alpha)$

The Q-function can be obtained by the antinormal-ordering characteristic function

$$Q(\alpha) = \int \frac{d^2\eta}{\pi^2} e^{\eta^*\alpha-\eta\alpha^*}\chi_A(\eta), \tag{3.19}$$

where the characteristic function $\chi_A(\eta)$ is

$$\chi_A(\eta) = \mathrm{Tr}\left[\rho e^{-\eta^* a}e^{\eta a^\dagger}\right]. \tag{3.20}$$

The ensemble average of an operator X in this representation is

$$\langle X\rangle = \int d^2\alpha\,\chi^A(\alpha)Q(\alpha), \tag{3.21}$$

where

$$\chi^A(\alpha) = \sum_{n,m}\chi^A_{nm}\alpha^n(\alpha^*)^m. \tag{3.22}$$

The coefficient χ^A_{nm} is the antinormal-ordering representation of an operator X,

$$X = \sum_{m,n}\chi^A_{nm}a^n(a^\dagger)^m. \tag{3.23}$$

Summary 2: Coherent States

(a) The phase space of photon states or ensembles is described by the two-dimensional complex α plane.

(b) The real part and imaginary part of α are related to the quadrature operator X and Y.

$$X = \mathrm{Re}[\alpha], \tag{3.24}$$
$$Y = \mathrm{Im}[\alpha]. \tag{3.25}$$

(c) A coherent state $|\alpha_0\rangle$ is a fuzzy circle on the complex α plane.

(d) The coherent states are not orthogonal, so they are overcomplete.

(e) There are three ways to write the probability density

- Wigner distribution $W(\alpha)$: symmetric ordering
- Q-function $Q(\alpha)$: antinormal ordering
- P-function $P(\alpha)$: normal ordering

4 Recent Development

- Gaussian quantum information, Gaussian State: [3]
- non-Gaussian quantum states, continuous-variable quantum systems, Wigner function, Wigner negativity: [4]
- quantum teleportation protocol for Gaussian states
- Encoding a qubit in an oscillator, GKP code
- cat states
- Circuit quantum electrodynamics, superconductor qubit, transmon qubit, microwave

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