Quantization of Fields

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Contents

In the modern framework of physics, quantum mechanics is a complete theory that becomes a classical theory in a macroscopic world. For example, $\mathbf{F} = m\mathbf{a}$ approximates quantum mechanics in the limit of large objects. However, the classical theories were discovered first, such as $\mathbf{F} = m\mathbf{a}$ and the Maxwell's equaitons. The quantum forms of these classical theories were not known. Physicists developed quantum theories from these classical theories. Such a process is called "**quantization**", which makes the new theory have discrete physical properties.

One of the quantization approaches is to find the canonical coordinates of classical theories and let them have a nonzero commutation relation. For example, $[x, p] = i\hbar$

The strategy to quantize fields is essentially the same as that for a harmonic oscillator. We think of electromagnetic modes as oscillations. Every mode with a specific frequency ω behaves as a harmonic oscillator.

1 Canonical Quantization

The steps to quantize a harmonic oscillator are summarized as follows

Note 1: Quantization of a Harmonic Oscillator

1. Find the canonical variables with the total energy quadratic in both variables. The Hamiltonian of a harmonic oscillators consists of canonical variables x and p.[a]

$$\text{total energy} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

2. Replace the classical variables x and p by $\hat{x}$ and $\hat{p}$ and obtain the Hamiltonian.

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2}$$

3. Impose the commutation relation

$$[\hat{x}, \hat{p}] = i\hbar$$

4. Make changes of variables to $\hat{a}$ and $\hat{a}^\dagger$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right),$$
$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right),$$

and obtain

$$\mathcal{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

[a] Canonical variables are initially from the classical mechanics. Classically, the canonical coordinate q and canonical momentum p satisfy the Poisson bracket relation. In canonical quantization, the Poisson bracket is replaced by the commutation relation.

The quantization of a particle in a quadratic potential inspired scientists to learn how to quantize other oscillations. For any other harmonic oscillations, the idea is first to find the canonical variables. For electromagnetic waves, we will use the analogies

$$\text{particle}: \quad x \sim a + a^\dagger, \quad p \sim -a + a^\dagger \tag{1.1}$$

$$\text{Light}: \quad \mathbf{E} \sim a + a^\dagger, \quad \mathbf{B} \sim -a + a^\dagger \tag{1.2}$$

Note 2: Physical meaning of a and $a^\dagger$

(a) a and $a^\dagger$ can be thought as the complex amplitudes of oscillation. a is the amplitude of a positive-frequency oscillation $e^{-i\omega t}$, and $a^\dagger$ is the amplitude of a negative-frequency oscillation $e^{i\omega t}$.

(b) If we ignore the coefficients, $x = (a + a^\dagger)/2$ and $p \sim (a - a^\dagger)/(2i)$ are indeed the real part and the imaginary part of the amplitude.

Note 3: Classical Mechanics

The Hamiltonian $\mathcal{H}(q_i, p_i)$ of a classical system can be written as a function of the canonical coordinates q_i and canonical momentums p_i. Canonical variables, by definition, satisfy the Poisson bracket

$$\{q_i, p_j\} = \delta_{ij}. \tag{1.3}$$

The definition of the Poisson bracket is

$$\{f, g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}. \tag{1.4}$$

The dynamical equations of a physical quantity A are given by

$$\frac{dA}{dt} = \{A, \mathcal{H}\}. \tag{1.5}$$

Using x and p as an example, the Hamiltonian can be written as $\mathcal{H} = \frac{p^2}{2m} + V(x)$. The equations of motions are given by Eq. (1.5),

$$\frac{dx}{dt} = \frac{p}{m}, \tag{1.6}$$

$$\frac{dp}{dt} = -\frac{\partial V}{\partial x}. \tag{1.7}$$

In canonical quantization, the Poisson brackets are replaced by the commutators.

2 Mode Functions As Canonical Operators

The Maxwell's equations in matter read

$$\nabla \cdot (\epsilon(\mathbf{r})\mathbf{E}) = 0 \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.3}$$

$$\nabla \times \mathbf{B} = \mu(\mathbf{r})\epsilon(\mathbf{r})\frac{\partial \mathbf{E}}{\partial t} \tag{2.4}$$

Since the Maxwell's equations are linear differential equations, to find the solution is indeed an eigenvalue problem. The eigenvalue is ω, and the the eigenmodes are

$$\mathbf{E}^c_\omega(\mathbf{r},t) = \boldsymbol{\mathcal{E}}_\omega(\mathbf{r})e^{-i\omega t}, \tag{2.5}$$

$$\mathbf{B}^c_\omega(\mathbf{r},t) = \boldsymbol{\mathcal{B}}_\omega(\mathbf{r})e^{-i\omega t}. \tag{2.6}$$

Here, $\boldsymbol{\mathcal{E}}_\omega$ and $\boldsymbol{\mathcal{B}}_\omega$ are complex functions, and the superscript c indicates that the field $\mathbf{E}^c_\omega(\mathbf{r},t)$ is a complex number. Later, we will use them to construct real mode functions. The dielectric function $\epsilon(\mathbf{r})$ and permeability $\mu(\mathbf{r})$ determine the field profiles of the mode functions. The total field is a Fourier integral of the mode functions.

$$\mathbf{E}(\mathbf{r},t) = \int_{-\infty}^{\infty} \alpha(\omega)\boldsymbol{\mathcal{E}}_\omega e^{-i\omega t}(\mathbf{r})d\omega, \tag{2.7}$$

where $\alpha(\omega)$ is the Fourier component.

2.1 Single Mode

For an electromagnetic mode of a frequency ω, we look for real solutions of the forms,

$$\mathbf{E}_\omega(\mathbf{r},t) = \boldsymbol{\mathcal{E}}_\omega(\mathbf{r})e^{-i\omega t} + \boldsymbol{\mathcal{E}}^*_\omega(\mathbf{r})e^{i\omega t} \tag{2.8}$$

$$\mathbf{B}_\omega(\mathbf{r},t) = \boldsymbol{\mathcal{B}}_\omega(\mathbf{r})e^{-i\omega t} + \boldsymbol{\mathcal{B}}^*_\omega(\mathbf{r})e^{i\omega t}, \tag{2.9}$$

which satisfies the Maxwell equations. The solutions to the $\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})$ and $\boldsymbol{\mathcal{B}}_\omega(\mathbf{r})$ will depend on the spatial arrangement of the $\epsilon(\mathbf{r})$ and $\mu(\mathbf{r})$. The complex field $\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})$ satisfies

$$\nabla \cdot (\epsilon(\mathbf{r})\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})) = 0, \tag{2.10}$$

$$\nabla \times (\nabla \times \boldsymbol{\mathcal{E}}_\omega(\mathbf{r})) = \mu(\mathbf{r})\epsilon(\mathbf{r})\omega^2\boldsymbol{\mathcal{E}}_\omega(\mathbf{r}). \tag{2.11}$$

One can solve the above equations analytically for simple geometries or numerically when geometries are more complicated. Once the $\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})$ is obtained, the magnetic field $\boldsymbol{\mathcal{B}}_\omega(\mathbf{r})$ is given by

$$\begin{aligned} \nabla \times \boldsymbol{\mathcal{E}}_\omega(\mathbf{r}) &= i\omega\boldsymbol{\mathcal{B}}_\omega(\mathbf{r}) \\ \Rightarrow \boldsymbol{\mathcal{B}}_\omega(\mathbf{r}) &= \frac{\nabla \times \boldsymbol{\mathcal{E}}_\omega(\mathbf{r})}{i\omega}. \end{aligned} \tag{2.12}$$

The total energy of the mode is

$$\mathcal{H}_\omega = \int dv \left(\frac{\epsilon(\mathbf{r}) E_\omega^2(\mathbf{r})}{2} + \frac{B_\omega^2(\mathbf{r})}{2\mu(\mathbf{r})} \right), \tag{2.13}$$

which is similar to

$$\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}. \tag{2.14}$$

with the analogies

$$x \sim \mathbf{E}_\omega(\mathbf{r}), \tag{2.15}$$
$$p \sim \mathbf{B}_\omega(\mathbf{r}). \tag{2.16}$$

It is natural to speculate[1] that

$$\mathbf{E}_\omega(\mathbf{r}) \sim \boldsymbol{\mathcal{E}}_\omega(\mathbf{r}) a + \boldsymbol{\mathcal{E}}_\omega^*(\mathbf{r}) a^\dagger, \tag{2.17}$$
$$\mathbf{B}_\omega(\mathbf{r}) \sim -\boldsymbol{\mathcal{B}}_\omega(\mathbf{r}) a + \boldsymbol{\mathcal{B}}_\omega^*(\mathbf{r}) a^\dagger. \tag{2.18}$$

We define the following field operators

$$\mathbf{E}_\omega(\mathbf{r}) = \frac{\left[\boldsymbol{\mathcal{E}}_\omega(\mathbf{r}) a + \boldsymbol{\mathcal{E}}_\omega^*(\mathbf{r}) a^\dagger\right]}{2}, \tag{2.19}$$
$$\mathbf{B}_\omega(\mathbf{r}) = \frac{i\left[-\boldsymbol{\mathcal{B}}_\omega(\mathbf{r}) a + \boldsymbol{\mathcal{B}}_\omega^*(\mathbf{r}) a^\dagger\right]}{2} \tag{2.20}$$

with the normalization conditions

$$\int dv \epsilon |\mathcal{E}_\omega(\mathbf{r})|^2 = \hbar\omega. \tag{2.21}$$

Plugging Eqs. (2.19) and (2.20) in Eq. (2.13), we obtain the Hamiltonian of a single electromagnetic mode,

$$\mathcal{H}_\omega = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right). \tag{2.22}$$

All the observables contain the creation and annihilation operator. We can first solve the dynamics of $a(t)$ and obtain all the dynamics. Using the Heisenberg's picture, the equation reads

$$\frac{\partial a}{\partial t} = \frac{i}{\hbar}[\mathcal{H}, a] \tag{2.23}$$
$$= -i\omega a, \tag{2.24}$$

which has the solution

$$a(t) = a(0) e^{-i\omega t}. \tag{2.25}$$

The operator $a^\dagger(t)$ is the hermitian conjugate of $a(t)$,

$$a^\dagger(t) = a^\dagger(0) e^{i\omega t}. \tag{2.26}$$

[1]You might have the same questions that I had as a student. What are the origins of using a harmonic model to quantize fields? Why is it valid? Why are **E** and **B** the canonical variables? I should say that, at least in my opinion, we can not **derive** physics from the first place. Typically, theorists would make educational guesses about the formulations. Such guesses are then to be examined by experiments. The validities rely on whether the results can explain the observations. To date, it is still the most consistent theory.

Derivation 1: Bonus Credits!

It requires some effort to derive Eq. (2.22). We sketch the steps

(a) Plug Eqs. (2.19) and (2.20) in Eq. (2.13).

(b) Show that the integral of the magnetic term is equivalent to the electric term. Replace the magnetic term with Eq. (2.12). Calculate the integrals with two curls by the integration by parts. Use the identity of vector calculus

$$\int_{\mathcal{V}} dv \mathbf{F} \cdot (\nabla \times \mathbf{A}) = \int_{\mathcal{V}} dv \mathbf{A} \cdot (\nabla \times \mathbf{F}) + \int_{\mathcal{S}} (\mathbf{A} \times \mathbf{F}) \cdot d\mathbf{a}, \tag{2.27}$$

where $\mathbf{A}$ and $\mathbf{F}$ are arbitrary vector fields. Use Eq. (2.11) to get rid of the curls.

(c) Use the normalization condition Eq. (2.21).

Note 4: Quantization for Fields

The procedures to quantize a field are:

(a) Find the two canonical variables where the total energy is quadratic in both variables. For example, let the two canonical variables be q and p.

(b) Impose the canonical commutation relation $[q, p] = i\hbar$.

(c) Define the creation and annihilation operators in terms of q and p such that $[a, a^\dagger] = 1$.

(d) Write the Hamiltonian in terms of a and $a^\dagger$.

Exercise 1: Quantization for LC circuit

Show that the total energy of an LC circuit is

$$E = \frac{\phi^2}{2L} + \frac{Q^2}{2C}, \tag{2.28}$$

where ϕ is the magnetic flux. The frequency ω of the LC oscillation is $\omega = \sqrt{1/LC}$, and

$$E = \frac{\phi^2}{2L} + \frac{L\omega^2 Q^2}{2}. \tag{2.29}$$

In this form, we have $L \sim m$, $\phi \sim p$, and $Q \sim x$. Thus, we enforce the relation

$$[\hat{Q}, \hat{\phi}] = i\hbar. \tag{2.30}$$

Check the units in the above equation are consistent. Find the a and $a^\dagger$ in terms of ϕ, Q, L, ω.

2.2 Multimode

We have shown how to quantize a single mode of light. We can extend the formulation to multimodes. Let m denote the quantum number of a mode. The total Hamiltonian is

$$\mathcal{H} = \sum_m \hbar\omega_m \left(a_m^\dagger a_m + \frac{1}{2} \right). \tag{2.31}$$

For example, m can denote the discrete quantum number of a waveguide or the continuous quantum number $\mathbf{k}$ of a plane wave. If m are discrete numbers, we have the relations

$$[a_m, a_{m'}^\dagger] = \delta_{mm'}. \tag{2.32}$$

The total field is

$$\mathbf{E}(\mathbf{r}) = \sum_m \mathbf{E}_m(\mathbf{r}). \tag{2.33}$$

The field operators of the mode m are

$$\mathbf{E}_m(\mathbf{r}) = \frac{\left[\boldsymbol{\mathcal{E}}_m(\mathbf{r})a + \boldsymbol{\mathcal{E}}_m^*(\mathbf{r})a^\dagger\right]}{2}, \tag{2.34}$$

$$\mathbf{B}_m(\mathbf{r}) = \frac{i\left[-\boldsymbol{\mathcal{B}}_m(\mathbf{r})a + \boldsymbol{\mathcal{B}}_m^*(\mathbf{r})a^\dagger\right]}{2} \tag{2.35}$$

with the normalization conditions

$$\int dv\epsilon|\mathcal{E}_m(\mathbf{r})|^2 = \hbar\omega_m. \tag{2.36}$$

The magnetic field operator is given by

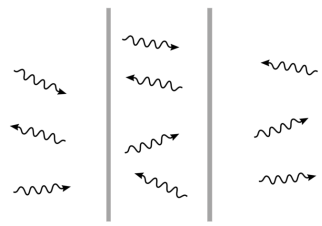
$$\boldsymbol{\mathcal{B}}_m(\mathbf{r}) = \frac{\nabla \times \boldsymbol{\mathcal{E}}_m(\mathbf{r})}{i\omega_m}. \tag{2.37}$$

Example 1: Casimir Force in a Nutshell!

The Casimir force, also known as the Casimir effect, is a physical force that arises from the quantum fluctuations of a field. This force was predicted by Dutch physicist Hendrik Casimir for electromagnetic systems in 1948. In classical theories, the ground state of a vacuum has zero electric field. However, in a quantum vacuum, the ground state energy is not zero since each allowed mode contributes $1/2\hbar\omega$. Thus, nonzero electric fields exist and produce pressure. That is, the vacuum is not empty!
The vacuum energy of the total Hamiltonian is

$$\left\langle 0 \left| \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2} \right) \right| 0 \right\rangle = \sum_{\mathbf{k}} \frac{\hbar\omega_{\mathbf{k}}}{2}. \tag{2.38}$$

The integral depends on how many modes there are. The most famous example is the Casimir effect. Consider two parallel metal plates.



The modes in the middle have the wave vector

$$\mathbf{k} = \left(\frac{N\pi}{d}, k_y, k_z\right). \tag{2.39}$$

Therefore, the vacuum energy of the middle space is

$$E_0(d) = \frac{\hbar}{2} \times 2 \times \left(\int \frac{L_y dk_y}{2\pi} \int \frac{L_z dk_z}{2\pi}\right) \sum_N c\sqrt{k_y^2 + k_z^2 + \frac{N^2\pi^2}{d^2}}. \tag{2.40}$$

This integral is divergent for any separation d. The potential energy of the system $U(d)$ is defined by

$$U(d) = E_0(\infty) - E_0(d). \tag{2.41}$$

Although both the two terms are divergent, their difference can be evaluated (See Ref. [1] or Sec. 2.6 of Ref. [2]) as

$$U(d) = \frac{-\pi^2\hbar c L_y L_z}{720 d^3}. \tag{2.42}$$

The force per unit area is then

$$\frac{F_c}{L_y L_z} = \frac{1}{L_y L_z}\frac{-\partial U}{\partial d} = -\frac{\pi^2\hbar c}{240 d^4}. \tag{2.43}$$

2.3 Number States (Fock States)

The eigenstates of the photon Hamiltonian, Eq. (2.31) are the direct product of the number states $|n_1\rangle \otimes |n_2\rangle$.... which is denoted as $|n_1 n_2 ...\rangle$. The total energy of the number states $|n_1 n_2 ...\rangle$ is

$$\langle ...n_2 n_1|\mathcal{H}|n_1 n_2...\rangle = \sum_m \left\langle ...n_2 n_1 \left| \hbar\omega_m \left(a_m^\dagger a_m + \frac{1}{2}\right) \right| n_1 n_2 ... \right\rangle \tag{2.44}$$

$$= \sum_m \left(n_m + \frac{1}{2}\right)\hbar\omega_m. \tag{2.45}$$

For simplicity, we consider a single-mode system in the following. Since the number states are the eigenstates. The expectation values of the observables are static. The expectation values of $\mathbf{E}(t)$ is

$$\langle \mathbf{E}(t) \rangle = \left\langle n \left| \frac{\left[\mathcal{E}_\omega(\mathbf{r}) a + \mathcal{E}^*_\omega(\mathbf{r}) a^\dagger \right]}{2} \right| n \right\rangle = 0. \tag{2.46}$$

The standard deviation of $\mathbf{E}(t)$ of a number state $|n\rangle$ does not vanish

$$\sigma(\mathbf{E}(t)) = \sqrt{\langle \mathbf{E}(t)^2 \rangle - \langle \mathbf{E}(t) \rangle^2} \tag{2.47}$$

$$= \sqrt{\langle \mathbf{E}(t)^2 \rangle} \tag{2.48}$$

$$= |\mathcal{E}_\omega(\mathbf{r})| \sqrt{\frac{n + \frac{1}{2}}{2}} \tag{2.49}$$

Exercise 2: Standard Deviation

Show Eq. (2.49). Hint: the operator $\mathbf{E}(t)^2$ is

$$\mathbf{E}(t)^2 = \left(\frac{\left[\mathcal{E}_\omega(\mathbf{r}) a + \mathcal{E}^*_\omega(\mathbf{r}) a^\dagger \right]}{2} \right)^2 \tag{2.50}$$

$$= \frac{|\mathcal{E}_\omega(\mathbf{r})|^2 (a a^\dagger + a^\dagger a) + \left[\mathcal{E}_\omega(\mathbf{r}) \cdot \mathcal{E}_\omega(\mathbf{r}) a^2 + \mathcal{E}^*_\omega(\mathbf{r}) \cdot \mathcal{E}^*_\omega(\mathbf{r}) (a^\dagger)^2 \right]}{4}. \tag{2.51}$$

The expectation of $\mathbf{E}(t)^2$ of a number state is

$$\left\langle n \middle| \mathbf{E}(t)^2 \middle| n \right\rangle. \tag{2.52}$$

2.4 Plane Waves

The eigenmodes in vacuum are the plane waves with the quantum number $\mathbf{k}$ and s (polarizations). The eigenmode $\mathcal{E}_m(\mathbf{r})$ is

$$\mathcal{E}_m(\mathbf{r}) = \mathcal{E}_{\mathbf{k},s}(\mathbf{r}) \tag{2.53}$$

$$= \frac{1}{\sqrt{V}} \mathcal{E}_{\mathbf{k},s} e^{i\mathbf{k}\cdot\mathbf{r}} \tag{2.54}$$

$$= \sqrt{\frac{\hbar\omega}{\epsilon_0 V}} \mathbf{e}_{\mathbf{k},s} e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{2.55}$$

where V is the volume where the waves exist. $\mathbf{e}_{\mathbf{k},s}$ denotes the two possible polarizations. The total Hamiltonian reads

$$\mathcal{H} = \sum_{\mathbf{k},s} \hbar\omega_{\mathbf{k}} \left(a^\dagger_{\mathbf{k},s} a_{\mathbf{k},s} + \frac{1}{2} \right). \tag{2.56}$$

The electric and magnetic field operators are

$$\mathbf{E}_{\mathbf{k},s}(\mathbf{r}) = \frac{\left[\mathcal{E}_{\mathbf{k},s}a + \mathcal{E}^*_{\mathbf{k},s}(\mathbf{r})a^\dagger\right]}{2}$$
$$= \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\epsilon_0 V}}\frac{\left[\mathbf{e}_{\mathbf{k},s}e^{i\mathbf{k}\cdot\mathbf{r}}a + \mathbf{e}^*_{\mathbf{k},s}e^{-i\mathbf{k}\cdot\mathbf{r}}a^\dagger\right]}{2}, \tag{2.57}$$

$$\mathbf{B}_{\mathbf{k},s}(\mathbf{r}) = \frac{\hat{k}}{c}\times\mathbf{E}_{\mathbf{k},s}$$
$$= \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\epsilon_0 V}}\frac{\left[\hat{k}\times\mathbf{e}_{\mathbf{k},s}e^{i\mathbf{k}\cdot\mathbf{r}}a + \hat{k}\times\mathbf{e}^*_{\mathbf{k},s}e^{-i\mathbf{k}\cdot\mathbf{r}}a^\dagger\right]}{2c}. \tag{2.58}$$

3 Thermal Ensemble

An ensemble of photons is specified by the density matrices. The most classic example is a system in the thermal equilibrium. The equilibrium is reached when a photonic system is in contact with a heat reservoir (environment). For a given temperature T, according to statistical mechanics, the probability of occupying a state n is proportional to

$$p(n) \sim e^{-\frac{E_n}{k_B T}}, \tag{3.1}$$

where k_B is the Boltzmann's constant. Considering the normalization, the probability is

$$p(n) = \frac{e^{-\frac{E_n}{k_B T}}}{\sum_m e^{-\frac{E_m}{k_B T}}} \tag{3.2}$$
$$= \frac{e^{-\frac{E_n}{k_B T}}}{Z}, \tag{3.3}$$

with the partition function Z

$$Z = \sum_m e^{-\frac{E_m}{k_B T}}. \tag{3.4}$$

Thus, the density operator of a thermal ensemble is

$$\rho_{\text{th}} = \sum_n p(n)|n\rangle\langle n| \tag{3.5}$$
$$= \frac{\sum_n e^{-\frac{E_n}{k_B T}}|n\rangle\langle n|}{Z} \tag{3.6}$$
$$= \frac{e^{-\frac{\mathcal{H}}{k_B T}}}{\text{Tr}[e^{-\frac{\mathcal{H}}{k_B T}}]} \tag{3.7}$$

Exercise 3: Partition Function

Show that the partition function Z of a single-mode photonic system is

$$Z = \frac{\exp\left(-\frac{\hbar\omega}{2k_B T}\right)}{1 - \exp\left(-\frac{\hbar\omega}{k_B T}\right)}. \tag{3.8}$$

Use $E_m = \left(m + \frac{1}{2}\right)\hbar\omega$ in Eq. (3.4)

The average number of the thermal ensemble is

$$\begin{aligned}
\langle \hat{N} \rangle &= \mathrm{Tr}[\rho_{\mathrm{th}} \hat{N}] &(3.9)\\
&= \sum_m \langle m | \rho_{\mathrm{th}} \hat{N} | m \rangle &(3.10)\\
&= \sum_m m \langle m | \rho_{\mathrm{th}} | m \rangle &(3.11)\\
&= \sum_{m,n} \frac{m e^{-\frac{\hbar\omega(n+1/2)}{k_B T}}}{Z} \langle m | n \rangle \langle n | m \rangle &(3.12)\\
&= \sum_m \frac{m e^{-\frac{\hbar\omega(m+1/2)}{k_B T}}}{Z} \quad \text{See Derivation 2} &(3.13)\\
&= \frac{1}{\exp \frac{\hbar\omega}{k_B T} - 1}, &(3.14)
\end{aligned}$$

which is the Bose-Einstein distribution.

Derivation 2: Trick of Sums of Series

Let

$$\tilde{Z}(x) = \sum_{m=0}^{\infty} e^{-mx} = \frac{1}{1 - e^{-x}}. \tag{3.15}$$

The trick to calculate the following sums

$$\tilde{Z}_l(x) \equiv \sum_{m=0}^{\infty} m^l e^{-mx}, \tag{3.16}$$

where l is an integer, is from the relation

$$\tilde{Z}_l(x) = (-1)^l \frac{\partial^l \tilde{Z}}{\partial x^l}. \tag{3.17}$$

Substituting Eq. (3.15) into (3.17) and doing the differentiation, you can obtain a closed form of the sum, Eq. (3.16).

Exercise 4: Standard Derivation of $\hat{N}$

Calculate $\sigma\left(\hat{N}\right)$ of a thermal ensemble of temperature T. Use

$$\sigma\left(\hat{N}\right) = \sqrt{\langle \hat{N}^2\rangle - \langle \hat{N}\rangle^2}, \tag{3.18}$$

$$\langle \hat{N}\rangle = \mathrm{Tr}[\rho_{\mathrm{th}}\hat{N}], \tag{3.19}$$

$$\langle \hat{N}^2\rangle = \mathrm{Tr}[\rho_{\mathrm{th}}\hat{N}^2]. \tag{3.20}$$

3.1 Black-Body Radiation

The average energy of one single mode is $\langle \hat{N}\rangle\hbar\omega$. The black-body radiation is defined as the radiation of a large enough thermally equilibrium system. Such a system has the properties

- The system is large enough so that the modes inside are the plane waves $\mathbf{E} = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}$.
- The system is thermal equilibrium with a well-defined temperature T.

Each $\mathbf{k}$ corresponds to two modes (left/right circular polarizations) for such a system. The total number of modes M (not photon number) is

$$M = 2\sum_{\mathbf{k}}. \tag{3.21}$$

But, the $\mathbf{k}$ becomes a continuous number when the system is vast. In the continuous limit, it becomes (see Derivation 3)

$$M = \frac{1}{\pi^2}\int_0^\infty k^2 dk. \tag{3.22}$$

This gives an infinitely large number since k has no upper bound. We can change the variable of the integral from k to ω by $\omega = ck$

$$M = \int_0^\infty \frac{\omega^2}{\pi^2 c^3} d\omega. \tag{3.23}$$

The M itself is not too meaningful. The number of modes within ω and $\omega + d\omega$ is more meaningful. This is called the density of state $g(\omega)$, given by

$$g(\omega) = \frac{\omega^2}{\pi^2 c^3}. \tag{3.24}$$

With this definition, the total number M

$$M = \int_0^\infty g(\omega) d\omega. \tag{3.25}$$

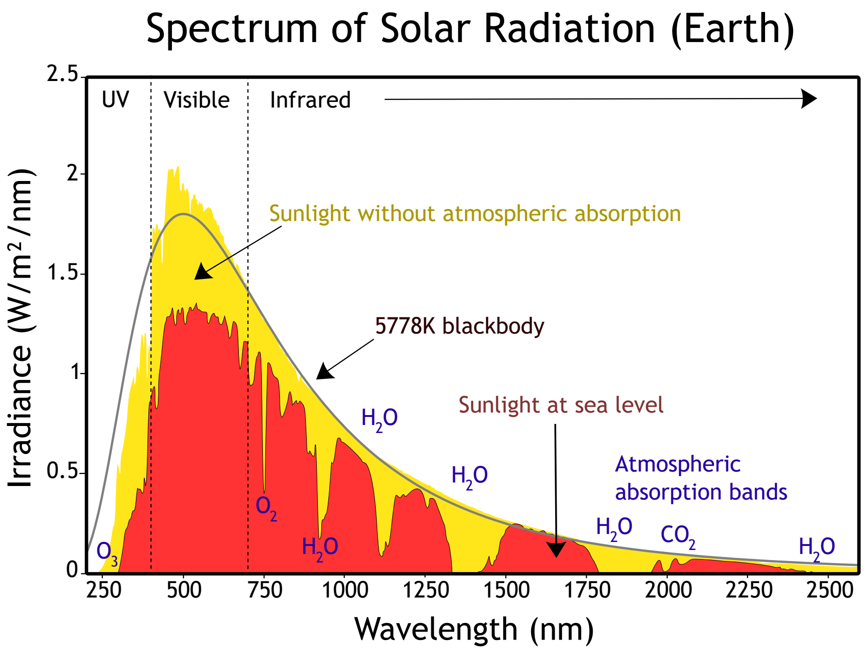


Figure 1: Energy density of a thermal ensemble of photons from Sun and the black-body radiation. (Picture credit: Wikimedia)

The density of state $g(\omega)$ is the number of modes per unit volume within ω and $\omega + d\omega$. The average energy density $U(\omega)$ (energy per unit volume) is then

$$U(\omega) = \langle \hat{N} \rangle g(\omega) \hbar\omega \tag{3.26}$$

$$= \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{\exp \frac{\hbar\omega}{k_B T} - 1}. \tag{3.27}$$

Its classical analog is the Rayleigh-Jeans formula

$$U_{\text{classical}}(\omega) = g(\omega) k_B T = \frac{\omega^2}{\pi^2 c^3} k_B T, \tag{3.28}$$

This leads to classical physics's ultraviolet catastrophe, i.e., the energy density diverges as $\omega \to \infty$. The total energy density U_{tot} is

$$U_{\text{tot}} = \int d\omega U(\omega) = \frac{\pi^2 k_B^4 T^4}{15 c^3 \hbar^3}. \tag{3.29}$$

The famous Stefan-Boltzmann law states that the power radiated by a heated object is proportional to T^4.

Derivation 3: Density of States

A cuboid has the side lengths L_x, L_y and L_z. We assume that the cuboid is large enough so that a plane wave $\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}$ can propagate in any direction. The system should satisfy the periodic boundary conditions so that the allowed wave vector $\mathbf{k} = (k_x, k_y, k_z)$ is

$$k_x = \frac{2\pi l_x}{L_x} \tag{3.30}$$

$$k_y = \frac{2\pi l_y}{L_y} \tag{3.31}$$

$$k_z = \frac{2\pi l_z}{L_z} \tag{3.32}$$

where l_x, l_y and l_z are integers. Note that l_x, l_y and l_z can be negative. The change of the total number m of modes is

$$\Delta m = 2\Delta l_x \Delta l_y \Delta l_z = 2\left(\frac{L_x L_y L_z}{(2\pi)^3}\right)\Delta k_x \Delta k_y \Delta k_z, \tag{3.33}$$

$$\Delta k_x \equiv \frac{2\pi}{L_x}, \tag{3.34}$$

$$\Delta k_y \equiv \frac{2\pi}{L_y}, \tag{3.35}$$

$$\Delta k_z \equiv \frac{2\pi}{L_z}. \tag{3.36}$$

where the factor 2 accounts for the polarizations. In the continuum limit, it becomes

$$\frac{dm}{V} = \left(\frac{1}{4\pi^3}\right) dk_x dk_y dk_z \tag{3.37}$$

$$= \frac{1}{4\pi^3} 4\pi k^2 dk \tag{3.38}$$

$$= \frac{1}{\pi^2}\frac{\omega^2 d\omega}{c^3}, \tag{3.39}$$

$$\Rightarrow g(\omega) \equiv \frac{1}{V}\frac{dm}{d\omega} = \frac{\omega^2}{\pi^2 c^3}. \tag{3.40}$$

Since l_x, l_y and l_z can be negative, it means that k_x, k_y and k_z can be negative. Hence the integral $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_x dk_y dk_z$ can be converted to $\int_0^{\infty} 4\pi k^2 dk$.

Note 5: Paradox: Density of States

As a smart student, you may wonder why we use the periodic boundary conditions and what if we use the vanishing boundary conditions so that instead of 2π, then the allowed wave vector $\mathbf{k} = (k_x, k_y, k_z)$ becomes

$$k_x = \frac{\pi l_x}{L_x}, \tag{3.41}$$

$$k_y = \frac{\pi l_y}{L_y}, \tag{3.42}$$

$$k_z = \frac{\pi l_z}{L_z}. \tag{3.43}$$

Would this difference lead to a different density of states $g(\omega)$? The answer is no. Of course, the density of states $g(\omega)$ should be the same no matter how one calculates it since there is only one physical truth. The resolution to this paradox is that for the vanishing boundary conditions, the modes are not plane waves but standing waves

$$\mathbf{E} \sim \sin(k_x L_x)\sin(k_y L_y)\sin(k_z L_z). \tag{3.44}$$

Since the modes do not propagate, the wave numbers are positive $k_x > 0$, $k_y > 0$, and $k_z > 0$ ($l_x > 0$, $l_y > 0$ and $l_z > 0$). Hence, the integrals over k_x, k_y, and k_z start from 0 to ∞. The integral $\int_0^\infty \int_0^\infty \int_0^\infty dk_x dk_y dk_z$ is now converted to $\frac{1}{8}\int_0^\infty 4\pi k^2 dk$. The $\frac{1}{8}$ accounts for that only the shell in the first octant is counted. So overall, you will obtain the same $g(\omega)$. **Actually, the $g(\omega)$ should be the same regardless of the boundaries if the system is large enough**.

4 Quadrature Operators

We have applied the idea of a harmonic oscillator to quantize fields. The canonical variables of a particle, x and p are numbers. Unlike a particle, the canonical operators of a photon, $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ are vector functions. In other words, to completely determine $\mathbf{E}(\mathbf{r})$, we have to know its value at every position $\mathbf{r}$. In contrast, x does not depend on other coordinates. Their similarities are the creation and annihilation operators a and $a^\dagger$. It is then useful to define the **dimensionless** operators for photons. We introduce the quadrature operators,

$$X = \frac{a + a^\dagger}{2}, \tag{4.1}$$

$$Y = \frac{a - a^\dagger}{2i}. \tag{4.2}$$

The operator, X, is the dimensionless position operator, and the operator, Y, is the dimensionless momentum. They have the relation

$$[X, Y] = \frac{i}{2}. \tag{4.3}$$

Using the generalized uncertainty relation, we obtain

$$\sigma(X)\sigma(Y) \geq \frac{|\langle [X, Y] \rangle|}{2} = \frac{1}{4}. \tag{4.4}$$

The electric field operator of a mode m is rewritten as

$$\mathbf{E}_m(\mathbf{r}) = \mathrm{Re}[\boldsymbol{\mathcal{E}}_m(\mathbf{r})]X - \mathrm{Im}[\boldsymbol{\mathcal{E}}_m(\mathbf{r})]Y. \tag{4.5}$$

In the case of plane waves, the electric field operator of a mode $\{\mathbf{k}, s\}$ is

$$\mathbf{E}_{\mathbf{k},s}(\mathbf{r}) = \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\epsilon_0 V}}\Big\{\mathrm{Re}[\mathbf{e}_{\mathbf{k},s}]\cos(\mathbf{k}\cdot\mathbf{r})X - \mathrm{Im}[\mathbf{e}^*_{\mathbf{k},s}]\sin(\mathbf{k}\cdot\mathbf{r})Y\Big\}. \tag{4.6}$$

5 Research Topics

The current note has a pedagogic order with fundamental assumptions, main equations, derivation, and some consequences. Such a structure is suitable for having a detailed understanding but may not be efficient for catching research topics. It is too late to access research topics after learning you have learned everything or the entire course. Many students think they must be well-prepared before reading a research paper. The reality is that it takes forever to be well-prepared. Even worse is that what you learned is outdated (of course, I will try to avoid this) or not applicable to current trends or your interests. The better strategy is (i) developing core concepts after one stage of learning and (ii) using these concepts to read papers. You may encounter many troubles from reading papers, but then know more about what you need.

For this reason, I summarize the main concepts in this note and list some current research keywords related to these concepts.

Concepts:

- Electromagnetic waves are quantum harmonic oscillators. Indeed, every wave in classical theories can be quantized. EM waves → photons, acoustic wave → phonons, ...
- Each mode is a harmonic oscillator.
- Zero-point energy gives quantum fluctuation.
- Numbers states can express any photon state.
- Temperature can determine the numbers in each mode.
- Quadrature operators are real and imaginary amplitudes of a mode. a and $a^\dagger$ are complex amplitudes.

Research keywords:

- single photon state: how to generate? In which physical systems? Applications?

- NOON state: an entangled state of two modes. Quantum lithography, quantum metrology.
- GHZ (Greenberger–Horne–Zeilinger state) state: an entangled state of three modes.
- super-Planckian emission: emission from a heated quantum system
- thermal emitter, superradiance
- active radiative cooling and heating
- photonic heat engine
- quantum thermal light
- LOQC (linear optical quantum computing)[3], PIC, silicon photonics
- FBQC (fusion-based quantum computation)
- How to make a qubit with photons? polarization, frequency bin, time bin, dual-rail, orbital angular momentum
- coherent Ising machines
- quadrature operators, phase space, quantum communication
- quantum communication with continuous variables

Exercise 5: Find and read a research paper

Use the above keywords from one of the items to search a paper after 2000 (or the year you were born) and more than 50 citations with a search engine like Google Scholar and Web of Science.

- Provide the reference (title, authors, journal, volume, page number).
- Read the paper with the method: https://web.stanford.edu/class/cs114/reading-keshav.pdf
- Rewrite the abstract for this paper by yourself. Make it brief.
- Summarize the main results and conclusions by listing them.
- Your opinions, questions, and comments on this paper.
- Future perspective.

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