

Optical Components and Measurements

Jhih-Sheng Wu

2023

Contents

1	Beam Splitters	2
1.1	Single Photon	3
1.2	Two-Photon Hong-Ou-Mandel effect	5
1.3	N-Photon	6
1.4	Coherent States	6

1 Beam Splitters

A beam splitter is an optical component which is partially transparent. An incident beam on a beam splitter is partially reflected and partially transmitted, and thus split into two beams. Classically, an incident beam with an amplitude A_1 is split into a reflected beam with the amplitude A_1 and a transmitted beam with the amplitude A_2 . The amplitudes are related by the coefficients of reflection and transmission,

$$A_2 = rA_1, \quad (1.1)$$

$$A_3 = tA_1. \quad (1.2)$$

We can also consider another case where a beam of the amplitude A_0 is incident on the other side of the beam splitter. In this case, the amplitudes are related by

$$A_2 = t'A_0, \quad (1.3)$$

$$A_3 = r'A_0. \quad (1.4)$$

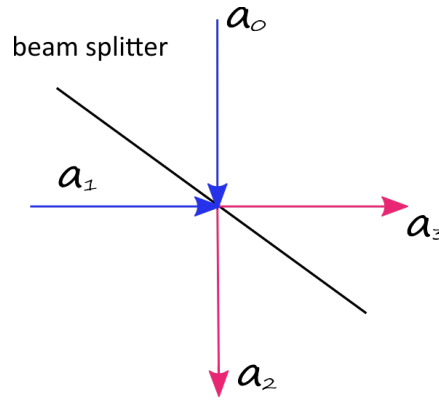


Figure 1: Beam splitter. Quantum descriptions where the amplitudes are replaced by the annihilation operators.

A beam splitter is described by the scattering matrix

$$\begin{pmatrix} A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} \equiv U \begin{pmatrix} A_0 \\ A_1 \end{pmatrix}. \quad (1.5)$$

Because of energy conservation, the matrix U must be unitary so that

$$|r|^2 + |t|^2 = 1 \quad (1.6)$$

$$|r'|^2 + |t'|^2 = 1 \quad (1.7)$$

$$tr^* + t'^*r' = 0. \quad (1.8)$$

In order to satisfy Eq. (1.8), the phase of each beam cannot be arbitrary. Let's consider the example of a symmetric beam splitter. Let the phase between r and t is θ , that is, $\text{Arg}\left[\frac{t}{r}\right] = \theta$. A symmetric beam splitter implies $\text{Arg}\left[\frac{t'}{r'}\right] = \theta$. Eq. (1.8) becomes

$$|r||t|(e^{i\theta} - e^{-i\theta}) = 0, \quad (1.9)$$

which gives $\theta = \pm \frac{\pi}{2}$. A discussion about the phases can be found in the Ref. [1].

The quantum description of a beam splitter is simply to replace amplitudes by annihilation operators. Let the right-going photons have the annihilation operator a_1 , and the bottom-going photons have the annihilation operator a_0 . A beam splitter is described by the scattering matrix

$$\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = U \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}, \quad (1.10)$$

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} t'^* & r' \\ r^* & t^* \end{pmatrix} \begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = U^\dagger \begin{pmatrix} a_2 \\ a_3 \end{pmatrix}. \quad (1.11)$$

The incident beams, a_0 and a_1 , are treated as two independent modes with the commutation relations

$$[a_0, a_0^\dagger] = 1, \quad (1.12)$$

$$[a_1, a_1^\dagger] = 1, \quad (1.13)$$

$$[a_0, a_1^\dagger] = 0. \quad (1.14)$$

From Eqs. (1.12), (1.13), (1.14), and (1.10), one can show that the operators a_2 and a_3 automatically satisfy

$$[a_i, a_j^\dagger] = \delta_{ij}. \quad (1.15)$$

Physically, these relations mean that after a beam splitter, a beam is split into two independent modes a_2 and a_3 .

Below, we are going to discuss what happens to a quantum light after passing a beam splitter. We will consider the cases of a single photon state, N -photon state, and a coherent state. We will see that the Fock states exhibit the quantum natures, where the output states are entangled, while the output state of a coherent state can be factorized.

1.1 Single Photon

The incident state $|0\rangle_0|1\rangle_1$ can be expressed as

$$|0\rangle_0|1\rangle_1 = a_1^\dagger |0\rangle_0|0\rangle_1. \quad (1.16)$$

From Eq. (1.11), the creation operators a_0^\dagger and a_1^\dagger are related to a_2^\dagger and a_3^\dagger by

$$\begin{pmatrix} a_0^\dagger \\ a_1^\dagger \end{pmatrix} = U^T \begin{pmatrix} a_2^\dagger \\ a_3^\dagger \end{pmatrix}. \quad (1.17)$$

After incidence on the beam splitter, the state becomes

$$U^T a_1^\dagger |0\rangle_0|0\rangle_1 = (ra_2^\dagger + ta_3^\dagger)|0\rangle_2|0\rangle_3 \quad (1.18)$$

$$= r|1\rangle_2|0\rangle_3 + t|0\rangle_2|1\rangle_3, \quad (1.19)$$

which is an entangled state.

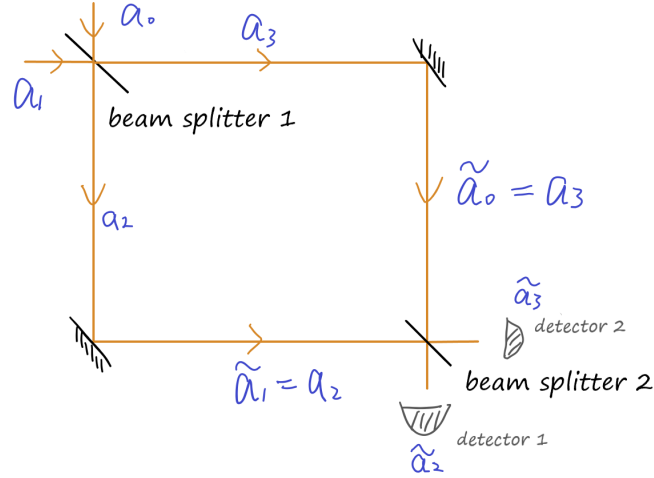


Figure 2: Mach-Zehnder interferometer.

Consider a Mach-Zehnder interferometer with two 50:50 beam splitters of $r = \frac{i}{\sqrt{2}}$ and $t = \frac{1}{\sqrt{2}}$. Let the initial state be $|0\rangle_0|1\rangle_1$. After the first beam splitter, the state becomes

$$\frac{i}{\sqrt{2}}|1\rangle_2|0\rangle_3 + \frac{1}{\sqrt{2}}|0\rangle_2|1\rangle_3. \quad (1.20)$$

When the state arrives at the second beam splitter, the state becomes

$$\frac{i}{\sqrt{2}}|1\rangle_2|0\rangle_3 + \frac{e^{i\theta}}{\sqrt{2}}|0\rangle_2|1\rangle_3, \quad (1.21)$$

where θ is a phase shift due to the difference of the two paths.

To deal with the second beam splitter, we first rename the modes. Modes 2 (a_2) and 3 (a_3) in the Eq. (1.21) become the incident beams to the second beam splitter. We rename Mode 2 (a_2) as the new Mode 1 (\tilde{a}_1) since it is right-going and Mode 3 (a_3) as the new Mode 0 (\tilde{a}_0) since it is bottom-going. For simplicity, we skip the tilde signs below. Now, we can use the same scattering matrix to find the final state after the second beam splitter,

$$\begin{pmatrix} a_2^\dagger \\ a_3^\dagger \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{e^{i\theta}}{\sqrt{2}} a_0^\dagger \\ \frac{i}{\sqrt{2}} a_1^\dagger \end{pmatrix} \quad (1.22)$$

$$= \begin{pmatrix} \frac{(e^{i\theta}-1)}{2} a_0^\dagger \\ \frac{i(e^{i\theta}+1)}{2} a_1^\dagger \end{pmatrix} \quad (1.23)$$

The probability at D_1 is $\left| \frac{(e^{i\theta}-1)}{2} \right|^2 = \sin^2 \frac{\theta}{2}$. The probability at D_2 is $\left| \frac{i(e^{i\theta}+1)}{2} \right|^2 = \cos^2 \frac{\theta}{2}$. This is a more rigorous description of a single-photon interference.

1.2 Two-Photon Hong-Ou-Mandel effect

The Hong-Ou-Mandel effect is a simple but novel phenomenon of quantum optics. It is an example where quantum interference leads to a non-classical result. Consider a symmetric beam 50-50 beam splitter. Now two single-photon states hit the beam splitter from the different sides, respectively. As shown in Fig. 3, the initial state is $|1\rangle_0|1\rangle_1$. What will be the output state? Since the total input photon number is two, the output state could be $|2\rangle_2|0\rangle_3$, $|0\rangle_2|2\rangle_3$, and $|1\rangle_2|1\rangle_3$. Here the subscripts of the ket denote on which side the photon state is, as in Fig. 3. Hence a general expression for the output state is the superposition of the above possible states,

$$|\text{output}\rangle = \alpha|2\rangle_2|0\rangle_3 + \beta|0\rangle_2|2\rangle_3 + \gamma|1\rangle_2|1\rangle_3. \quad (1.24)$$

From the classical point of view, the state $|1\rangle_2|1\rangle_3$ seems the most possible state. However, in 1987, Hong et al., showed that for a symmetric beam splitter, $\gamma = 0$, that is, the probability of the state $|1\rangle_2|1\rangle_3$ is zero.[2] Specifically, the output state is

$$|\text{output}\rangle = U^T a_0^\dagger |0\rangle U^T a_1^\dagger |0\rangle \quad (1.25)$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} a_0^\dagger \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ a_1^\dagger \end{pmatrix} |0\rangle \quad (1.26)$$

$$= \frac{1}{2} (a_2^\dagger + i a_3^\dagger) (i a_2^\dagger + a_3^\dagger) |0\rangle \quad (1.27)$$

$$= \frac{i}{2} [(a_2^\dagger)^2 + (a_3^\dagger)^2] |0\rangle \quad (1.28)$$

$$= \frac{i}{\sqrt{2}} (|2\rangle_2|0\rangle_3 + |0\rangle_2|2\rangle_3) \quad (1.29)$$

Surprisingly, two identical single photons are bunched after hitting a beam splitter from different sides. In the linear optical quantum computing, the Hong-Ou-Mandel effect is the mechanism for the logic gates.

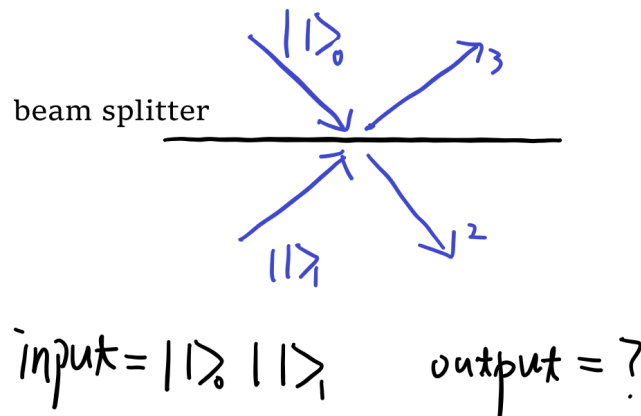


Figure 3: Hong-Ou-Mandel effect.

1.3 N-Photon

Let the initial state be $|0\rangle_0|N\rangle_1 = \frac{(a_1^\dagger)^N}{\sqrt{N!}}|0\rangle$. After a beam splitter, the state becomes

$$\frac{(Ua_1^\dagger)^N}{\sqrt{N!}}|0\rangle = \frac{(ta_2^\dagger + ra_3^\dagger)^N}{\sqrt{N!}}|0\rangle. \quad (1.30)$$

1.4 Coherent States

Let the initial state be $|0\rangle_0|\alpha\rangle_1 = D_1[\alpha]|0\rangle = e^{\alpha a_1^\dagger - \alpha^* a_1}|0\rangle$. After a beam splitter, the state becomes

$$e^{\alpha Ua_1^\dagger - \alpha^* U^\dagger a_1}|0\rangle = e^{\alpha ta_2^\dagger - \alpha^* t^* a_2} e^{\alpha ra_3^\dagger - \alpha^* r^* a_3}|0\rangle \quad (1.31)$$

$$= D_2[t\alpha]D_3[r\alpha]|0\rangle \quad (1.32)$$

$$= |t\alpha\rangle_2|r\alpha\rangle_3. \quad (1.33)$$

The input of a coherent state is split into a product of two coherent states. Unlike the single-photon case, this state is not entangled.

Consider a Mach-Zehnder interferometer with two 50:50 beam splitters of $r = \frac{i}{\sqrt{2}}$ and $t = \frac{1}{\sqrt{2}}$. Let the initial state be $|0\rangle_0|\alpha\rangle_1$. After the first beam splitter, the state becomes

$$\left| \frac{i\alpha}{\sqrt{2}} \right\rangle_2 \left| \frac{\alpha}{\sqrt{2}} \right\rangle_3. \quad (1.34)$$

When the state arrives at the second beam splitter, the state becomes

$$\left| \frac{i\alpha}{\sqrt{2}} \right\rangle_2 \left| \frac{e^{i\theta}\alpha}{\sqrt{2}} \right\rangle_3. \quad (1.35)$$

where θ is a phase shift due to the difference of the two paths. The final state after the second beam splitter is (see the figure: a_2 is the new a_1 and a_3 is the new a_0)

$$\left| \frac{(e^{i\theta} - 1)}{2} \right\rangle_2 \left| \frac{i(e^{i\theta} + 1)}{2} \right\rangle_3. \quad (1.36)$$

The intensity at D_1 is $|\alpha|^2 \left| \frac{e^{i\theta} - 1}{2} \right|^2 = \sin^2 \frac{\theta}{2} |\alpha|^2$. The intensity at D_2 is $|\alpha|^2 \left| \frac{e^{i\theta} + 1}{2} \right|^2 = \cos^2 \frac{\theta}{2} |\alpha|^2$. The two output beams are both coherent states. Thus, the phase θ can be obtained by

$$\frac{I_2 - I_1}{|\alpha|^2} = \cos \theta. \quad (1.37)$$

However, the amplitudes have uncertainty $\sigma(n) = \sqrt{\bar{n}}$. Thus, the phase has the uncertainty $\sigma(\theta) \sim \frac{1}{\sqrt{\bar{n}}}$. In experiments, we would like to use a coherent light (laser) with a well defined phase and a strong intensity such that the uncertainty in phase is small. But a strong-intensity light may lead to more noise such as radiation pressures, thermal noises, and so on. To solve this dilemma, lights with small $\sigma(n)$ is used. These lights are non-classical lights.

References

- [1] Z. Y. Ou and L. Mandel, Derivation of reciprocity relations for a beam splitter from energy balance, *American Journal of Physics* 57, 66 (1989)
- [2] C. K. Hong; Z. Y. Ou & L. Mandel . "Measurement of subpicosecond time intervals between two photons by interference". *Phys. Rev. Lett.* 59 (18): 2044-2046 (1987)