Non-Classical Light

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Contents

1 Correlation Functions

A correlation function is a mathematical tool used to measure the dependence between two or more variables. In physics, these variables are physical quantities depending on positions and time. For example, let $s_1(\mathbf{r},t)$ and and $s_2(\mathbf{r},t)$ be the two amplitudes of two scalar waves. The first order correlation function is

$$C^{(1)}(\Delta\mathbf{r},\Delta t) = \langle s_1^*(\mathbf{r}_1,t_1)s_2(\mathbf{r}_2,t_2)\rangle, \tag{1.1}$$

where $\langle\ldots\rangle$ denotes an ensemble average.

The correlation functions are used to describe spatial and temporal coherences of waves. The superposition of two waves is

$$|s_1(\mathbf{r}_1,t_1)+s_2(\mathbf{r}_2,t_2)|^2 = |s_1(\mathbf{r}_1,t_1)|^2 + |s_2(\mathbf{r}_2,t_2)| + 2\mathrm{Re}[s_1(\mathbf{r}_1,t_1)s_2^*(\mathbf{r}_2,t_2)]. \tag{1.2}$$

The ensemble-averaged interference is

$$\langle 2\mathrm{Re}[s_1(\mathbf{r}_1,t_1)s_2^*(\mathbf{r}_2,t_2)]\rangle = 2\mathrm{Re}[C(\Delta\mathbf{r},\Delta t)]. \tag{1.3}$$

Correlation functions are called auto-correlation functions if s_1 and s_2 are the same variables. If s_1 is the same source as s_2 and $\Delta\mathbf{r} = 0$, the correlation functions measure the temporal coherence. If s_1 is the same source as s_2 and $\Delta t = 0$, the correlation functions measure the spatial coherence. We can define the dimensionless correlations function $g^{(1)}$, [1] called the first-order correlation function or normalized correlation function,

$$g^{(1)}(\Delta\mathbf{r},\Delta t) = \frac{\langle s_1^*(\mathbf{r}_1,t_1)s_2(\mathbf{r}_2,t_2)\rangle}{\sqrt{\langle|s_1(\mathbf{r}_1,t_1)|^2\rangle\langle|s_2(\mathbf{r}_2,t_2)|^2\rangle}}. \tag{1.4}$$

According to the Schwartz inequality, $0 \le g^{(1)} \le 1$. The coherences are related to $g^{(1)}$ by

$$|g^{(1)}| = 1, \quad \text{coherent,} \tag{1.5}$$

$$0 < |g^{(1)}| < 1, \quad \text{partially coherent,} \tag{1.6}$$

$$|g^{(1)}| = 0, \quad \text{completely incoherent.} \tag{1.7}$$

The coherence function $g^{(1)}$ typically decreases at time goes or traveled optical length increases. The process is called decoherence. Two main sources of decoherence are (a) non-monochromatic light and (b) noises due to collisions or scatterings. We might model the decoherence as

$$g^{(1)}(t) = g^{(1)}(0)\exp\left(-\frac{t}{\tau_c}\right), \tag{1.8}$$

where τ_c is the coherence time. If a light source is not monochromatic and has a band width $\Delta\omega$, the coherent time is $\tau_c \sim \frac{1}{\Delta\omega}$.

[1] In the literature, people use $\gamma^{(1)}$ for classical cases and $g^{(1)}$ for quantum cases. Here, I use $g^{(1)}$ for both the cases.

In theories of probability and statistics, we can specify a probability distribution of a variable X if we know all the moments of X, i.e., $\langle X\rangle$, $\langle X^2\rangle$, $\langle X^3\rangle$,... If we want to fully specify the relation between X and Y, we need to know not only $\langle XY\rangle$ but also the higher order terms such as $\langle X^2Y^2\rangle$, $\langle X^3Y^3\rangle$. One can define the high-order autocorrelation functions are defined as

$$C^{(2)}(x_1,x_2,x_3,x_4) = \langle s^*(x_1)s^*(x_2)s(x_3)s(x_4)\rangle \tag{1.9}$$

ans so on. Here, x_n denotes $(\mathbf{r}_n, t_n)$. One useful case is

$$C^{(2)}(x_1,x_2,x_2,x_1) = \langle I(x_1)I(x_2)\rangle, \tag{1.10}$$

which is called the intensity-intensity correlation function. The order of x_n in the correlation function does not matter for classical fields, but as we will see soon, the order for quantum fields does matter. The second-order coherence function $g^{(2)}$ is defined as

$$g^{(2)}(x_2-x_1) \equiv g^{(2)}(x_1,x_2,x_2,x_1) = \frac{C^{(2)}(x_1,x_2,x_2,x_1)}{C^{(1)}(x_1,x_1)C^{(1)}(x_2,x_2)}. \tag{1.11}$$

In quantum optics, waves are electromagnetic fields. The scalar field $s(x)$ is replaced by $\mathcal{E} \equiv \boldsymbol{\mathcal{E}}\cdot\hat{e}$, where $\hat{e}$ is a unit vector. For example, the first-order coherence function becomes

$$g^{(1)}(\Delta\mathbf{r},\Delta t) = \frac{\langle \mathcal{E}_1^*(\mathbf{r}_1,t_1)\mathcal{E}_2(\mathbf{r}_2,t_2)\rangle}{\sqrt{\langle|\mathcal{E}_1(\mathbf{r}_1,t_1)|^2\rangle\langle|\mathcal{E}_2(\mathbf{r}_2,t_2)|^2\rangle}}. \tag{1.12}$$

1.1 Definitions in Quantum Optics

When defining correlation functions for quantum optics, we have the following term

$$\langle \mathcal{E}_1^*(\mathbf{r}_1,t_1)\mathcal{E}_2(\mathbf{r}_2,t_2)\rangle. \tag{1.13}$$

Classically, the order of the product in the average does not matter. But quantumly, we have to deal with the order carefully. Physically, correlations are measured quantities. Measurement processes consist of absorptions of photons by the detectors. Say, first the detector absorbed one photon at t_1 and another photon at a latter time t_2. This process is described by two annihilation operators

$$a(t_2)a(t_1)|i\rangle, \tag{1.14}$$

where $|i\rangle$ is the initial state. The probability of the process is proportional to the norm of Eq. (1.14).

$$\langle i|a^\dagger(t_1)a^\dagger(t_2)a(t_2)a(t_1)|i\rangle. \tag{1.15}$$

Thus, we have the following summary

- All annihilation operators are on the right.
- All creation operators are on the left.

- An annihilation operator at an earlier time is on the right.
- An creation operator at an earlier time is on the left.

The correlation functions are given by the density-matrix approach

$$
\begin{aligned}
C^{(1)} &= \mathrm{Tr}[\rho \mathcal{E}^*(t_2) a^\dagger(t_2) \mathcal{E}(t_1) a(t_1)] && (1.16)\\
&= \mathcal{E}^*(t_2)\mathcal{E}(t_1)\mathrm{Tr}[\rho a^\dagger(t_2) a(t_1)], && (1.17)\\
C^{(2)} &= \mathrm{Tr}[\rho \mathcal{E}^*(t_1) a^\dagger(t_1) \mathcal{E}^*(t_2) a^\dagger(t_2) \mathcal{E}(t_2) a(t_2) \mathcal{E}(t_1) a(t_1)] && (1.18)\\
&= \mathcal{E}^*(t_1)\mathcal{E}^*(t_2)\mathcal{E}(t_2)\mathcal{E}(t_1)\mathrm{Tr}[\rho a^\dagger(t_1) a^\dagger(t_2) a(t_2) a(t_1)] && (1.19)\\
&= \tilde{I}(t_1)\tilde{I}(t_2)\mathrm{Tr}[\rho a^\dagger(t_1) a^\dagger(t_2) a(t_2) a(t_1)]. && (1.20)
\end{aligned}
$$

We should use these correlation function to calculate the quantum coherence functions defined in Eqs. (1.11) and (1.12).

1.2 Young's Interference and First-Order Coherent Function

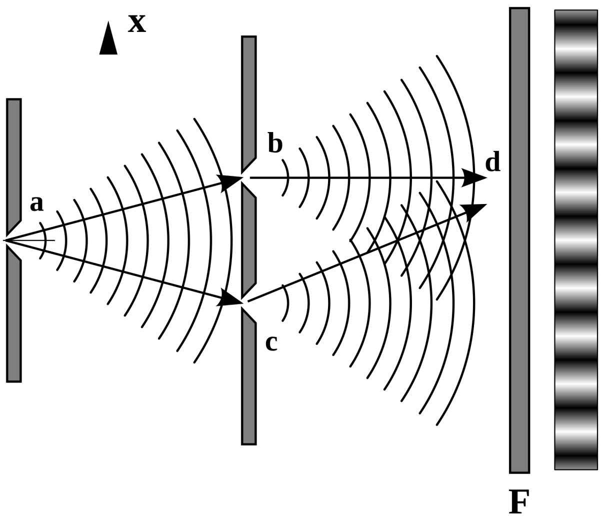

Figure 1: Young's interference.

Let the source generated single-mode photons whose annihilation operator is a. The photons then pass the two slits. The two slits are regarded as the light sources whose annihilation operators are a_1 and a_2. We assume the two slits are equal such that

$$
a = \frac{1}{\sqrt{2}}(a_1 + a_2). \tag{1.21}
$$

The intensity on the screen is indeed the first-order correlation function

$$I(t) = \tilde{I}(t)\mathrm{Tr}[\rho a^\dagger(t)a(t)] \tag{1.22}$$

$$= \frac{\tilde{I}(t)}{2}\left\{\mathrm{Tr}[\rho a_1^\dagger(t)a_1(t)] + \mathrm{Tr}[\rho a_2^\dagger(t)a_2(t)] + \mathrm{Tr}[\rho a_1^\dagger(t)a_2(t)] + \mathrm{Tr}[\rho a_2^\dagger(t)a_1(t)]\right\} \tag{1.23}$$

$$= \frac{\tilde{I}(t)}{2}\left\{\mathrm{Tr}[\rho a_1^\dagger(0)a_1(0)] + \mathrm{Tr}[\rho a_2^\dagger(0)a_2(0)] + \mathrm{Tr}[\rho a_1^\dagger(0)a_2(0)]e^{i\Phi} + \mathrm{Tr}[\rho a_2^\dagger(0)a_1(0)]e^{-i\Phi}\right\}, \tag{1.24}$$

where Φ is the phase difference due to the optical length. When the incident light is a one-photon state, we have the state after two slits

$$a^\dagger|0\rangle = \frac{1}{\sqrt{2}}\left(a_1^\dagger + a_2^\dagger\right)|0\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle). \tag{1.25}$$

The first-order correlation function is

$$I(t) = \tilde{I}(t)\left(\frac{1+\cos\Phi}{2}\right). \tag{1.26}$$

When the incident light is a two-photon state, we have the state after two slits

$$(a^\dagger)^2|0\rangle = \frac{1}{2}\left(a_1^\dagger + a_2^\dagger\right)^2|0\rangle = \frac{1}{2}(|20\rangle + \sqrt{2}|11\rangle + |02\rangle). \tag{1.27}$$

The first-order correlation function is

$$I(t) = 2\tilde{I}(t)\left(\frac{1+\cos\Phi}{2}\right). \tag{1.28}$$

For a n-photon state, we have

$$I(t) = n\tilde{I}(t)\left(\frac{1+\cos\Phi}{2}\right). \tag{1.29}$$

It can be shown that for a coherent state, we have

$$I(t) = \bar{n}\tilde{I}(t)\left(\frac{1+\cos\Phi}{2}\right). \tag{1.30}$$

We have the following notes

- Interference occurs even when for a single-photon state.
- First-order correlation functions describe interferences.
- First-order correlation functions **can not** distinguish what kind of photon state the light is. For number states and coherent states, the interferences are the same as $\cos\Phi$.

1.3 Hanbury Brown and Twiss Experiment and Second-Order Coherent Function

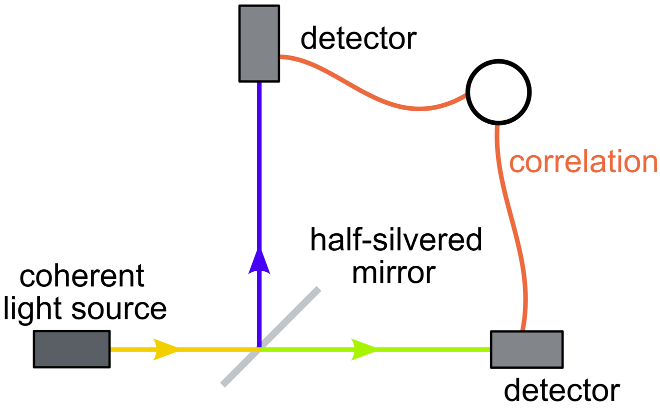

Figure 2: Hanbury Brown and Twiss experiment.

First-order coherence in Young's experiment can determine how monochromatic a light is, or noise levels by measuring coherent lengths. The properties revealed by first-order coherence are related to the modes and the environments (noise). However, first-order coherence does not tell which photon state it is, that is, if two states are from the same mode, first-order coherence can not distinguish their photon distribution. Say, to distinguish a number state $|n\rangle$ and a coherent state $|\alpha\rangle$ from the same mode, we have to use the second-order coherence function $g^{(2)}$.

In the 1950s, Hanbury Brown and Twiss developed an experiment to measure intensity-intensity correlations (see Fig. 2). The main difference from a Young's interference is that there are two intensity detectors instead of one. The two optical paths after the beam splitter lead to a time delay τ. The rates of coincident events is measured by the coincidence counter and proportional to

$$C^{(2)} = \langle \mathcal{E}^*(t_1)a^\dagger(t_1)\mathcal{E}^*(t_2)a^\dagger(t_2).\mathcal{E}(t_2)a(t_2)\mathcal{E}(t_1)a(t_1)\rangle \tag{1.31}$$

1.3.1 Classical Regime

Classically, we can write the second-order correlation function as

$$C^{(2)} = \langle I(t+\tau)I(t)\rangle, \tag{1.32}$$

and the second-order coherent function $g^{(2)}$ as

$$g^{(2)} = \frac{\langle I(t+\tau)I(t)\rangle}{\langle I(t+\tau)\rangle\langle I(t)\rangle}. \tag{1.33}$$

For a stationary light, $\langle I(t)\rangle$ is time-independent. The second-order coherent function $g^{(2)}$ becomes

$$g^{(2)} = \frac{\langle I(t+\tau)I(t)\rangle}{\langle I(t)\rangle^2}. \tag{1.34}$$

The following properties can be shown (only in the classical regime),

$$1 \leq g^{(2)}(0) < \infty, \tag{1.35}$$

$$g^{(2)}(\tau) \leq g^{(2)}(0). \tag{1.36}$$

For a chaotic light source, it can be shown that

$$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2, \tag{1.37}$$

and if $g^{(1)}(\tau) = e^{-\frac{\tau}{\tau_c}}$,

$$g^{(2)}(\tau) = 1 + e^{-\frac{2\tau}{\tau_c}}. \tag{1.38}$$

For chaotic lights, it is shown $g^{(2)}(0) = 2$. This is called the photon bunching-effects.

1.3.2 Quantum Regime

The orders of operators in the correlation functions should be treated carefully. For a single mode, the second-order coherent function is

$$g^{(2)}(\tau) = \frac{\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\rangle}{\langle a^\dagger(t+\tau)a(t+\tau)\rangle\langle a^\dagger(t)a(t)\rangle} \tag{1.39}$$

$$= \frac{\langle a^\dagger a^\dagger aa\rangle}{\langle a^\dagger a\rangle^2} \tag{1.40}$$

$$= 1 + \frac{\sigma^2(n) - \bar{n}}{\bar{n}^2} \tag{1.41}$$

We can show that

$$g^{(2)}(0) = \begin{cases} 2, & \text{thermal}, \\ 1, & \text{coherent}, \end{cases} \tag{1.42}$$

and for a number state

$$g^{(2)}(0) = \begin{cases} 0, & n = 0,\ 1, \\ \frac{n-1}{n}, & \text{else}. \end{cases} \tag{1.43}$$

As time increases, light becomes incoherent. We may use the result for chaotic light,

$$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2, \tag{1.44}$$

and $g^{(1)}(\tau) \to 0$ as $\tau \to \infty$. Thus as $\tau \to \infty$, $g^{(2)}(\tau) = 1$.

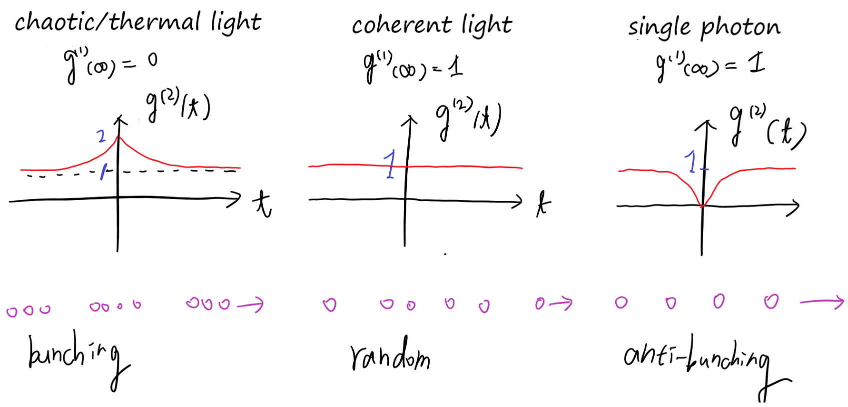

Figure 3: $g^{(2)}(\tau)$ of various photon ensembles.

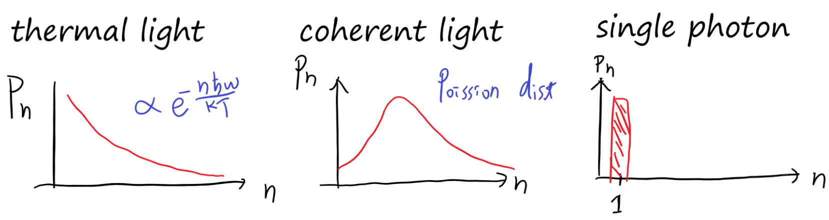

Figure 4: Photon counting statistics of various photon ensembles.

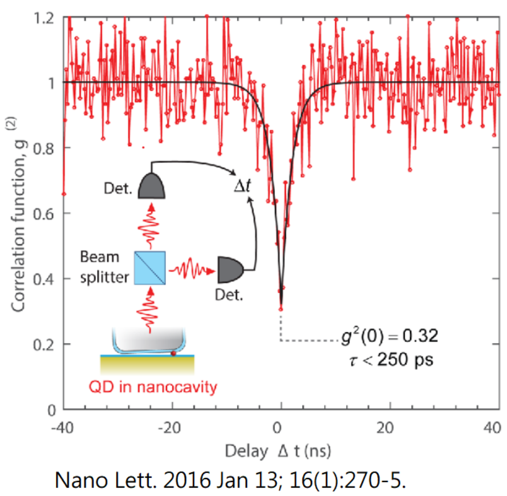

Figure 5: Experiment of single photon measurement.

Exercise 1: $g^{(2)}(0)$

Show that

(a)

$$g^{(2)}(0) = \frac{\langle a^\dagger a^\dagger a a\rangle}{\langle a^\dagger a\rangle^2} = 1 + \frac{\sigma^2(n) - \bar{n}}{\bar{n}^2}$$

(b) For a single mode thermal field, $g^{(2)}(0) = 2$.

Exercise 2: Coherence Functions

Calculate $g^{(2)}(0)$ for the following cases:

(a) a state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle). \tag{1.45}$$

(b) an ensemble

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|. \tag{1.46}$$

(c) an ensemble

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| + \frac{i}{2}|0\rangle\langle 1| + \frac{-i}{2}|1\rangle\langle 0| \tag{1.47}$$

Hint: use

$$g^{(2)}(0) = \frac{\langle a^\dagger a^\dagger a a\rangle}{\langle a^\dagger a\rangle^2} \tag{1.48}$$

Summary 1: Correlations of Light

1. Coherent light has $g^{(1)}(t) = 1$ and $g^{(2)}(t) = 1$. The photon number distribution is a Poisson distribution.

2. A chaotic light has a $g^{(1)} = 0$. There are many kinds of chaotic lights, for example, thermal lights, unpolarized lights, and so on. A chaotic light has $g^{(2)}(0) = 2$, which is the photon bunching. One can distinguish different kinds of chaotic lights by photon counting statistics. The p_n of a thermal light is proportional to $e^{-\frac{n\hbar\omega}{kT}}$. In general, p_n of a chaotic light can be arbitrary. It is possible that p_n of a chaotic light is a Poisson distribution. Hence, a Poisson distribution of p_n does not conclude that the light is coherent.

3. A single photon state has $g^{(1)}(t) = 1$ and $g^{(2)}(0) = 0$, which is a result

of the photon anti-bunching.

2 Quadrature Squeezing

The quadrature operators X and Y, satisfy

$$[X,Y] = \frac{i}{2} \tag{2.1}$$
$$\Rightarrow \sigma(X)\sigma(Y) \geq \frac{1}{4}. \tag{2.2}$$

The coherent states satisfy the minimum uncertainty equations,

$$\sigma(X)\sigma(Y) = \frac{1}{4} \tag{2.3}$$

and

$$\sigma(X) = \sigma(Y) = \frac{1}{2}. \tag{2.4}$$

which is a circle in the phase space. The conditions of a quadrature squeezing are

$$\sigma(X) < \frac{1}{2} \text{ or } \sigma(Y) < \frac{1}{2} \tag{2.5}$$

while keeping $\sigma(X)\sigma(Y) = \frac{1}{4}$. Pictorially, a squeezed state is an ellipse in the phase space with a area $\frac{\pi}{16}$. Of course, we can squeeze a state in any direction other than X or Y. We can define the rotated quadrature operator as the following

$$X'(\theta) = \frac{ae^{-i\theta} + a^\dagger e^{i\theta}}{2}, \tag{2.6}$$
$$Y'(\theta) = \frac{ae^{-i\theta} - a^\dagger e^{i\theta}}{2i}, \tag{2.7}$$

which represent a coordinate transform of the quadrature operators. Depending on the squeezed axis, we have the following squeezed states. Question: which one has the minimum uncertainty of the photon number?

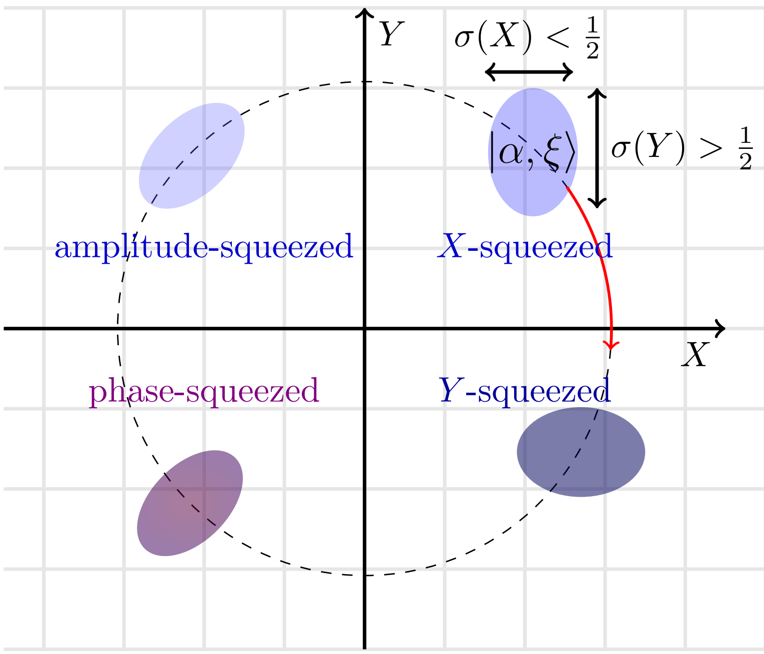

Figure 6: Squeezed States.

2.1 Squeezed Operators

Mathematically, a coherent state is generated by shifting a vacuum state in the phase space. This is done by the displacement operator $D(\alpha)$,

$$|\alpha\rangle = D(\alpha)|0\rangle. \tag{2.8}$$

We have shown that $D(\alpha)$ is the evolution operator U of a oscillating current source, that is, such a source creates a coherent state.

A squeezed state is generated by a squeeze operator,

$$S(\xi) = \exp\left(\frac{\xi^* a^2 - \xi(a^\dagger)^2}{2}\right), \tag{2.9}$$

where $\xi = re^{i\theta}$, and r is the squeeze parameter. A squeezed operator is a unitary operator. In principle, a unitary operator correspond a physical process. Observing the quadratic terms of the creation and annihilation operators, it is straightforward to speculate that **the physical processes are nonlinear**. This is because the quadratic terms come from the square of the electric field operators, $\mathbf{E}^2 = \left(\frac{\mathcal{E}a + \mathcal{E}^* a^\dagger}{2}\right)^2$. Squeeze operators have the relations

$$S^\dagger(\xi)S(\xi) = \mathbb{1}, \tag{2.10}$$

$$S^\dagger(\xi)aS(\xi) = a\cosh r - a^\dagger e^{i\theta}\sinh r, \tag{2.11}$$

$$S^\dagger(\xi)a^2S(\xi) = \left(a\cosh r - a^\dagger e^{i\theta}\sinh r\right)^2, \tag{2.12}$$

$$S^\dagger(\xi)a^\dagger S(\xi) = a^\dagger\cosh r - ae^{-i\theta}\sinh r, \tag{2.13}$$

$$S^\dagger(\xi)(a^\dagger)^2S(\xi) = \left(a^\dagger\cosh r - ae^{-i\theta}\sinh r\right)^2. \tag{2.14}$$

Let's first consider the squeezing of a vacuum state $S(\xi)|0\rangle$. The uncertainty of the squeezed state is

$$\sigma(X) = \frac{1}{2}\sqrt{\cosh^2 r + \sinh^2 r - 2\sinh r \cosh r \cos\theta}, \tag{2.15}$$

$$\sigma(Y) = \frac{1}{2}\sqrt{\cosh^2 r + \sinh^2 r + 2\sinh r \cosh r \cos\theta}. \tag{2.16}$$

When $\theta = 0$,

$$\sigma(X) = \frac{1}{2}e^{-r}, \tag{2.17}$$

$$\sigma(Y) = \frac{1}{2}e^{r}. \tag{2.18}$$

The state $S(\xi)|0\rangle$ is called the squeezed vacuum state, which the expectation value of the electric field is zero. We can obtain a more general squeeze state by applying both $D(\alpha)$ and $S(\xi)$ on a vacuum state,

$$|\alpha, \xi\rangle \equiv D(\alpha)S(\xi)|0\rangle. \tag{2.19}$$

Displacement operators have the relations

$$D^\dagger(\alpha)aD(\alpha) = a + \alpha, \tag{2.20}$$

$$D^\dagger(\alpha)a^\dagger D(\alpha) = a^\dagger + \alpha^*, \tag{2.21}$$

which add constants and do not change $\sigma(a)$ and $\sigma(a^\dagger)$. This means that $S(\xi)|0\rangle$ and $D(\alpha)S(\xi)|0\rangle$ have the same shapes in the phase space.

2.2 Number-State Representations

Let $|\xi\rangle = |0, \xi\rangle$ expressed in the number basis,

$$|\xi\rangle = \sum_n C_n |n\rangle., \tag{2.22}$$

where

$$C_n = \begin{cases} 0, & \text{odd}, \\ \frac{i^n}{\sqrt{\cosh r}} \frac{\sqrt{n!}}{2^{n/2}\left(\frac{n}{2}\right)!} e^{in\theta/2} \tanh^{n/2} r, & \text{even}. \end{cases} \tag{2.23}$$

For a general squeezed state, $|\alpha, \xi\rangle$, the coefficients are

$$C_n = \exp\left[-\frac{1}{2}|\alpha^2| - \frac{1}{2}(\alpha^*)^2 e^{i\theta}\tanh r\right] \frac{\left(\frac{e^{i\theta}\tanh r}{2}\right)^{n/2}}{\sqrt{n!\cosh r}} H_n\left[\frac{\alpha + \alpha^* e^{i\theta}\tanh r}{\sqrt{2e^{i\theta}\tanh r}}\right]. \tag{2.24}$$

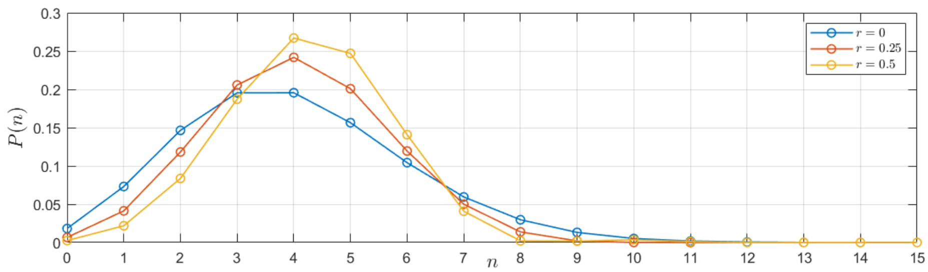

Figure 7: Photon counting of squeezed states. $\alpha = 2$

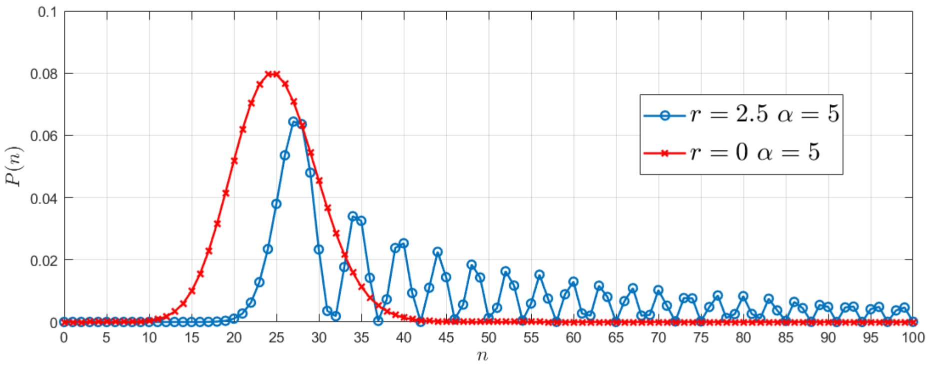

Figure 8: Photon counting of squeezed states. $\alpha = 5$