Quantization of Fields

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Contents

The strategy to quantize fields is essentially the same as that for a harmonic oscillator. We think electromagnetic modes as some sorts of oscillations. Every mode with a specific frequency ω behaves as a harmonic oscillator.

1 Canonical Quantization

The steps to quantize a harmonic oscillator are summarized as following

Note 1: Quantization of a Harmonic Oscillator

1. Find the canonical variables with what the total energy is quadratic in the both variables. The Hamiltonian of a harmonic oscillators consists of canonical variables x and p.[a]

$$\text{total energy} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

2. Replace the classical variables x and p by $\hat{x}$ and $\hat{p}$ and obtain the Hamiltonian.

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2}$$

3. Impose the commutation relation

$$[\hat{x}, \hat{p}] = i\hbar$$

4. Make changes of variables to $\hat{a}$ and $\hat{a}^\dagger$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right),$$
$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right),$$

and obtain

$$\mathcal{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

[a] Canonical variables are originally from the classical mechanics. Classically, the canonical coordinate q and canonical momentum p satisfy the Poisson bracket relation. In canonical quantization, the Poisson bracket is replaced by the commutation relation.

The quantization of a particle in a quadratic potential inspired scientists how to quantize other oscillations. For any other harmonic oscillations, the idea is to first find the canonical variables. For electromagnetic waves, we will use the analogies

$$\text{particle}: \quad x \sim a + a^\dagger, \quad p \sim -a + a^\dagger \tag{1.1}$$
$$\text{Light}: \quad \mathbf{E} \sim a + a^\dagger, \quad \mathbf{B} \sim -a + a^\dagger \tag{1.2}$$

Note 2: Physical meaning of a and $a^\dagger$

(a) a and $a^\dagger$ can be thought as the complex amplitudes of oscillation. a is the amplitude of a positive-frequency oscillation $e^{-i\omega t}$, and a is the amplitude of a negative-frequency oscillation $e^{i\omega t}$.

(b) If we ignore the coefficients, $x = (a + a^\dagger)/2$ and $p \sim (a - a^\dagger)/(2i)$ are indeed the real part and the imaginary part of the amplitude.

Note 3: Classical Mechanics

The Hamiltonian $\mathcal{H}(q_i, p_i)$ of a classical system can be written as a function of the canonical coordinates q_i and canonical momentums p_i. Canonical variables by definition satisfy the Poisson bracket

$$\{q_i, p_j\} = \delta_{ij}. \tag{1.3}$$

The definition of the Poisson bracket is

$$\{f, g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}. \tag{1.4}$$

The dynamical equations of a physical quantity A is given by

$$\frac{dA}{dt} = \{A, \mathcal{H}\}. \tag{1.5}$$

Using x and p as an example, the Hamiltonian can be written as $\mathcal{H} = \frac{p^2}{2m} + V(x)$. The equations of motions are given by Eq. (1.5),

$$\frac{dx}{dt} = \frac{p}{m}, \tag{1.6}$$
$$\frac{dp}{dt} = -\frac{\partial V}{\partial x}. \tag{1.7}$$

In canonical quantization, the Poisson brackets are replaced by the commutators.

2 Mode Functions As Canonical Operators

The Maxwell's equations in matter read

$$\nabla \cdot (\epsilon(\mathbf{r})\mathbf{E}) = 0 \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.3}$$

$$\nabla \times \mathbf{B} = \mu(\mathbf{r})\epsilon(\mathbf{r})\frac{\partial \mathbf{E}}{\partial t} \tag{2.4}$$

Since the Maxwell's equations are linear differential equations, to find the solution is indeed an eigenvalue problem. The eigenvalue is ω, and the the eigenmodes are

$$\mathbf{E}^c_\omega(\mathbf{r}, t) = \boldsymbol{\mathcal{E}}_\omega(\mathbf{r})e^{-i\omega t}, \tag{2.5}$$

$$\mathbf{B}^c_\omega(\mathbf{r}, t) = \boldsymbol{\mathcal{B}}_\omega(\mathbf{r})e^{-i\omega t}. \tag{2.6}$$

Here, $\boldsymbol{\mathcal{E}}_\omega$ and $\boldsymbol{\mathcal{B}}_\omega$ are complex functions, and the superscript c indicates that the field $\mathbf{E}^c_\omega(\mathbf{r}, t)$ is a complex number. Later, we will use them to construct real mode functions. The dielectric function $\epsilon(\mathbf{r})$ and permeability $\mu(\mathbf{r})$ determine the field profiles of the mode functions. The total field is a Fourier integral of the mode functions.

$$\mathbf{E}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \alpha(\omega)\boldsymbol{\mathcal{E}}_\omega e^{-i\omega t}(\mathbf{r})d\omega, \tag{2.7}$$

where $\alpha(\omega)$ is the Fourier component.

2.1 Single Mode

For an electromagnetic mode of a frequency ω, we look for real solutions of the forms,

$$\mathbf{E}_\omega(\mathbf{r}, t) = \boldsymbol{\mathcal{E}}_\omega(\mathbf{r})e^{-i\omega t} + \boldsymbol{\mathcal{E}}^*_\omega(\mathbf{r})e^{i\omega t} \tag{2.8}$$

$$\mathbf{B}_\omega(\mathbf{r}, t) = \boldsymbol{\mathcal{B}}_\omega(\mathbf{r})e^{-i\omega t} + \boldsymbol{\mathcal{B}}^*_\omega(\mathbf{r})e^{i\omega t}, \tag{2.9}$$

which satisfy the Maxwell equations. The solutions to the $\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})$ and $\boldsymbol{\mathcal{B}}_\omega(\mathbf{r})$ will depend on the spatial arrangement of the $\epsilon(\mathbf{r})$ and $\mu(\mathbf{r})$. The complex field $\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})$ satisfies

$$\nabla \cdot (\epsilon(\mathbf{r})\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})) = 0, \tag{2.10}$$

$$\nabla \times (\nabla \times \boldsymbol{\mathcal{E}}_\omega(\mathbf{r})) = \mu(\mathbf{r})\epsilon(\mathbf{r})\omega^2\boldsymbol{\mathcal{E}}_\omega(\mathbf{r}). \tag{2.11}$$

One can solve the above equations analytically for simple geometries or numerically when geometries are more complicated. Once the $\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})$ is obtained, the magnetic field $\boldsymbol{\mathcal{B}}_\omega(\mathbf{r})$ is given by

$$\nabla \times \boldsymbol{\mathcal{E}}_\omega(\mathbf{r}) = i\omega\boldsymbol{\mathcal{B}}_\omega(\mathbf{r})$$

$$\Rightarrow \boldsymbol{\mathcal{B}}_\omega(\mathbf{r}) = \frac{\nabla \times \boldsymbol{\mathcal{E}}_\omega(\mathbf{r})}{i\omega}. \tag{2.12}$$

The total energy of the mode is

$$\mathcal{H}_\omega = \int dv \left(\frac{\epsilon(\mathbf{r}) E_\omega^2(\mathbf{r})}{2} + \frac{B_\omega^2(\mathbf{r})}{2\mu(\mathbf{r})} \right), \tag{2.13}$$

which is similar to

$$\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}. \tag{2.14}$$

with the analogies

$$x \sim \mathbf{E}_\omega(\mathbf{r}), \tag{2.15}$$

$$p \sim \mathbf{B}_\omega(\mathbf{r}). \tag{2.16}$$

It is naturally to speculate[1] that

$$\mathbf{E}_\omega(\mathbf{r}) \sim \boldsymbol{\mathcal{E}}_\omega(\mathbf{r}) a + \boldsymbol{\mathcal{E}}_\omega^*(\mathbf{r}) a^\dagger, \tag{2.17}$$

$$\mathbf{B}_\omega(\mathbf{r}) \sim -\boldsymbol{\mathcal{B}}_\omega(\mathbf{r}) a + \boldsymbol{\mathcal{B}}_\omega^*(\mathbf{r}) a^\dagger. \tag{2.18}$$

We define the following field operators

$$\mathbf{E}_\omega(\mathbf{r}) = \frac{\left[\boldsymbol{\mathcal{E}}_\omega(\mathbf{r}) a + \boldsymbol{\mathcal{E}}_\omega^*(\mathbf{r}) a^\dagger\right]}{2}, \tag{2.19}$$

$$\mathbf{B}_\omega(\mathbf{r}) = \frac{i\left[-\boldsymbol{\mathcal{B}}_\omega(\mathbf{r}) a + \boldsymbol{\mathcal{B}}_\omega^*(\mathbf{r}) a^\dagger\right]}{2} \tag{2.20}$$

with the normalization conditions

$$\int dv \epsilon |\mathcal{E}_\omega(\mathbf{r})|^2 = \hbar\omega. \tag{2.21}$$

Plugging Eqs. (2.19) and (2.20) in Eq. (2.13), we obtain the Hamiltonian of a single electromagnetic mode,

$$\mathcal{H}_\omega = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right). \tag{2.22}$$

All the observables contains the creation and annihilation operator. We can first solve the dynamics of $a(t)$ and obtain all the dynamics. Using the Heisenberg's picture, the equation reads

$$\frac{\partial a}{\partial t} = \frac{i}{\hbar}[\mathcal{H}, a] \tag{2.23}$$

$$= -i\omega a, \tag{2.24}$$

which has the solution

$$a(t) = a(0) e^{-i\omega t}. \tag{2.25}$$

The operator $a^\dagger(t)$ is the hermitian conjugate of $a(t)$,

$$a^\dagger(t) = a^\dagger(0) e^{i\omega t}. \tag{2.26}$$

[1]You might have the same questions that I had as a student. What are the origins of using a harmonic model to quantize fields? Why is it valid? Why are **E** and **B** the canonical variables? I should say that at least in my opinion, we can not **derive** physics from the first place. Typically, theorists would make educational guesses about the formulations. Such guesses are then to be examined by experiments. The validities rely on whether the results can explain the observations. To date, it is still the most consistent theory.

Derivation 1: Bonus Credits!

It requires some efforts to derive Eq. (2.22). We sketch the steps

(a) Plug Eqs. (2.19) and (2.20) in Eq. (2.13).

(b) Show that the integral of the magnetic term is equivalent to the electric term. Replace the magnetic term with Eq. (2.12). Calculate the integrals with two curls by the integration by parts. Use the identity of vector calculus

$$\int_{\mathcal{V}} dv \mathbf{F} \cdot (\nabla \times \mathbf{A}) = \int_{\mathcal{V}} dv \mathbf{A} \cdot (\nabla \times \mathbf{F}) + \int_{\mathcal{S}} (\mathbf{A} \times \mathbf{F}) \cdot d\mathbf{a}, \tag{2.27}$$

where $\mathbf{A}$ and $\mathbf{F}$ are arbitrary vector fields. Use Eq. (2.11) to get rid of the curls.

(c) Use the normalization condition Eq. (2.21).

Note 4: Quantization for Fields

The procedures to quantize a field are:

(a) Find the two canonical variables, where the total energy is both quadratic in both variables. For example, let the two canonical variables be q and p.

(b) Impose the canonical commutation relation $[q, p] = i\hbar$.

(c) Define the creation and annihilation operators in terms of q and p such that $[a, a^\dagger] = 1$.

(d) Write the Hamiltonian in terms of a and $a^\dagger$.

Exercise 1: Quantization for LC circuit

Show that the total energy of an LC circuit is

$$E = \frac{\phi^2}{2L} + \frac{Q^2}{2C}, \tag{2.28}$$

where ϕ is the magnetic flux. The frequency ω of the LC oscillation is $\omega = \sqrt{1/LC}$, and

$$E = \frac{\phi^2}{2L} + \frac{L\omega^2 Q^2}{2}. \tag{2.29}$$

In this form, we have $L \sim m$, $\phi \sim x$, and $Q \sim p$. Thus, we enforce the relation

$$[\hat{\phi}, \hat{Q}] = i\hbar. \tag{2.30}$$

Check the units in the above equation are consistent. Find the a and $a^\dagger$ in terms of ϕ, Q, L, ω.

2.2 Multimode

We have shown how to quantize a single mode of light. We can extend the formulation to multimodes. Let m denote the quantum number of a mode. The total Hamiltonian is

$$\mathcal{H} = \sum_m \hbar\omega_m \left(a_m^\dagger a_m + \frac{1}{2}\right). \tag{2.31}$$

For example, m can denote the discrete quantum number of a waveguide, or the continuous quantum number $\mathbf{k}$ of a plane wave. If m are discrete numbers, we have the relations

$$[a_m, a_{m'}^\dagger] = \delta_{mm'}. \tag{2.32}$$

The total field is

$$\mathbf{E}(\mathbf{r}) = \sum_m \mathbf{E}_m(\mathbf{r}). \tag{2.33}$$

The field operators of the mode m are

$$\mathbf{E}_m(\mathbf{r}) = \frac{\left[\boldsymbol{\mathcal{E}}_m(\mathbf{r})a + \boldsymbol{\mathcal{E}}_m^*(\mathbf{r})a^\dagger\right]}{2}, \tag{2.34}$$

$$\mathbf{B}_m(\mathbf{r}) = \frac{i\left[-\boldsymbol{\mathcal{B}}_m(\mathbf{r})a + \boldsymbol{\mathcal{B}}_m^*(\mathbf{r})a^\dagger\right]}{2} \tag{2.35}$$

with the normalization conditions

$$\int dv \epsilon |\mathcal{E}_m(\mathbf{r})|^2 = \hbar\omega_m. \tag{2.36}$$

The magnetic field operator is given by

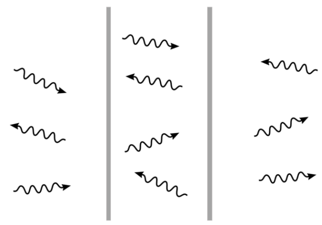
$$\boldsymbol{\mathcal{B}}_m(\mathbf{r}) = \frac{\nabla \times \boldsymbol{\mathcal{E}}_m(\mathbf{r})}{i\omega_m}. \tag{2.37}$$

Example 1: Casimir Force in a Nutshell!

The vacuum energy of the total Hamiltonian is

$$\left\langle 0 \middle| \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2}\right) \middle| 0 \right\rangle = \sum_{\mathbf{k}} \frac{\hbar\omega_{\mathbf{k}}}{2}. \tag{2.38}$$

The integral depends on how many modes there are. The most famous example is the Casimir effect. Consider two parallel metal plates.



The modes in the middle have the wave vector

$$\mathbf{k} = \left(\frac{N\pi}{d}, k_y, k_z\right). \tag{2.39}$$

Therefore, the vacuum energy of the middle space is

$$E_0(d) = \frac{\hbar}{2} \times 2 \times \left(\int \frac{L_y dk_y}{2\pi} \int \frac{L_z dk_z}{2\pi}\right) \sum_N c\sqrt{k_y^2 + k_z^2 + \frac{N^2\pi^2}{d^2}}. \tag{2.40}$$

This integral is divergent for any separation d. The potential energy of the system $U(d)$ is defined by

$$U(d) = E_0(\infty) - E_0(d). \tag{2.41}$$

Although both the two terms are divergent, their difference can be evaluated (See Ref. [1] or Sec. 2.6 of Ref. [2]) as

$$U(d) = \frac{-\pi^2\hbar c L_y L_z}{720 d^3}. \tag{2.42}$$

The force per unit area is then

$$\frac{F_c}{L_y L_z} = \frac{1}{L_y L_z}\frac{-\partial U}{\partial d} = -\frac{\pi^2\hbar c}{240 d^4}. \tag{2.43}$$

2.3 Number States (Fock States)

The eigenstates of the photon Hamiltonian, Eq. (2.31) are the direct product of the number states $|n_1\rangle \otimes |n_2\rangle$.... which is denoted as $|n_1 n_2 ...\rangle$. The total energy of the number states $|n_1 n_2 ...\rangle$ is

$$\langle ... n_2 n_1 | \mathcal{H} | n_1 n_2 ...\rangle = \sum_m \left\langle ... n_2 n_1 \middle| \hbar\omega_m \left(a_m^\dagger a_m + \frac{1}{2}\right) \middle| n_1 n_2 ... \right\rangle \tag{2.44}$$

$$= \sum_m \left(n_m + \frac{1}{2}\right)\hbar\omega_m. \tag{2.45}$$

For simplicity, we consider a single-mode system in the following. Since the number states are the eigenstates. The expectation values of the observables are static. The expectation values of $\mathbf{E}(t)$ is

$$\langle \mathbf{E}(t) \rangle = \left\langle n \left| \frac{\left[\mathcal{E}_\omega(\mathbf{r})a + \mathcal{E}^*_\omega(\mathbf{r})a^\dagger\right]}{2} \right| n \right\rangle = 0. \tag{2.46}$$

The standard deviation of $\mathbf{E}(t)$ of a number state $|n\rangle$ does not vanish

$$\sigma(\mathbf{E}(t)) = \sqrt{\langle \mathbf{E}(t)^2 \rangle - \langle \mathbf{E}(t) \rangle^2} \tag{2.47}$$
$$= \sqrt{\langle \mathbf{E}(t)^2 \rangle} \tag{2.48}$$
$$= |\mathcal{E}_\omega(\mathbf{r})| \sqrt{\frac{n + \frac{1}{2}}{2}} \tag{2.49}$$

Exercise 2: Standard Deviation

Show Eq. (2.49). Hint: the operator $\mathbf{E}(t)^2$ is

$$\mathbf{E}(t)^2 = \left(\frac{\left[\mathcal{E}_\omega(\mathbf{r})a + \mathcal{E}^*_\omega(\mathbf{r})a^\dagger\right]}{2} \right)^2 \tag{2.50}$$
$$= \frac{|\mathcal{E}_\omega(\mathbf{r})|^2 (aa^\dagger + a^\dagger a) + \left[\mathcal{E}_\omega(\mathbf{r}) \cdot \mathcal{E}_\omega(\mathbf{r}) a^2 + \mathcal{E}^*_\omega(\mathbf{r}) \cdot \mathcal{E}^*_\omega(\mathbf{r}) (a^\dagger)^2\right]}{4}. \tag{2.51}$$

The expectation of $\mathbf{E}(t)^2$ of a number state is

$$\left\langle n \left| \mathbf{E}(t)^2 \right| n \right\rangle. \tag{2.52}$$

2.4 Plane Waves

The eigenmodes in vacuum are the plane waves with the quantum number $\mathbf{k}$ and s (polarizations). The eigenmode $\mathcal{E}_m(\mathbf{r})$ is

$$\mathcal{E}_m(\mathbf{r}) = \mathcal{E}_{\mathbf{k},s}(\mathbf{r}) \tag{2.53}$$
$$= \frac{1}{\sqrt{V}} \mathcal{E}_{\mathbf{k},s} e^{i\mathbf{k}\cdot\mathbf{r}} \tag{2.54}$$
$$= \sqrt{\frac{\hbar\omega}{\epsilon_0 V}} \mathbf{e}_{\mathbf{k},s} e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{2.55}$$

where V is the volume where the waves exist. $\mathbf{e}_{\mathbf{k},s}$ denotes the two possible polarizations. The total Hamiltonian reads

$$\mathcal{H} = \sum_{\mathbf{k},s} \hbar\omega_{\mathbf{k}} \left(a^\dagger_{\mathbf{k},s} a_{\mathbf{k},s} + \frac{1}{2} \right). \tag{2.56}$$

The electric and magnetic field operators are

$$\mathbf{E}_{\mathbf{k},s}(\mathbf{r}) = \frac{\left[\mathcal{E}_{\mathbf{k},s} a + \mathcal{E}^*_{\mathbf{k},s}(\mathbf{r}) a^\dagger\right]}{2}$$
$$= \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\epsilon_0 V}} \frac{\left[\mathbf{e}_{\mathbf{k},s} e^{i\mathbf{k}\cdot\mathbf{r}} a + \mathbf{e}^*_{\mathbf{k},s} e^{-i\mathbf{k}\cdot\mathbf{r}} a^\dagger\right]}{2}, \tag{2.57}$$
$$\mathbf{B}_{\mathbf{k},s}(\mathbf{r}) = \frac{\hat{k}}{c} \times \mathbf{E}_{\mathbf{k},s}$$
$$= \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\epsilon_0 V}} \frac{\left[\hat{k} \times \mathbf{e}_{\mathbf{k},s} e^{i\mathbf{k}\cdot\mathbf{r}} a + \hat{k} \times \mathbf{e}^*_{\mathbf{k},s} e^{-i\mathbf{k}\cdot\mathbf{r}} a^\dagger\right]}{2c}. \tag{2.58}$$

3 Thermal Ensemble

An ensemble of photons is specified by the density matrices. The most classic example is a system in the thermal equilibrium. The equilibrium is reached when a photonic system is in contact with a heat reservoir (environment). For a given temperature T, according to statistical mechanics, the probability to occupy a state n is proportional to

$$p(n) \sim e^{-\frac{E_n}{k_B T}}, \tag{3.1}$$

where k_B is the Boltzmann's constant. Considering the normalization, the probability is

$$p(n) = \frac{e^{-\frac{E_n}{k_B T}}}{\sum_m e^{-\frac{E_m}{k_B T}}} \tag{3.2}$$
$$= \frac{e^{-\frac{E_n}{k_B T}}}{Z}, \tag{3.3}$$

with the partition function Z

$$Z = \sum_m e^{-\frac{E_m}{k_B T}}. \tag{3.4}$$

Thus, the density operator of a thermal ensemble is

$$\rho_{\text{th}} = \sum_n p(n)|n\rangle\langle n| \tag{3.5}$$
$$= \frac{\sum_n e^{-\frac{E_n}{k_B T}}|n\rangle\langle n|}{Z} \tag{3.6}$$
$$= \frac{e^{-\frac{\mathcal{H}}{k_B T}}}{\text{Tr}[e^{-\frac{\mathcal{H}}{k_B T}}]} \tag{3.7}$$

Exercise 3: Partition Function

Show that the partition function Z of a single mode photonic system is

$$Z = \frac{\exp\left(-\frac{\hbar\omega}{2k_BT}\right)}{1-\exp\left(-\frac{\hbar\omega}{k_BT}\right)}. \tag{3.8}$$

Use $E_m = \left(m+\frac{1}{2}\right)\hbar\omega$ in Eq. (3.4)

The average number of the thermal ensemble is

$$\begin{aligned}
\langle\hat{N}\rangle &= \mathrm{Tr}[\rho_{\mathrm{th}}\hat{N}] && (3.9)\\
&= \sum_m \langle m|\rho_{\mathrm{th}}\hat{N}|m\rangle && (3.10)\\
&= \sum_m m\langle m|\rho_{\mathrm{th}}|m\rangle && (3.11)\\
&= \sum_{m,n} \frac{me^{-\frac{\hbar\omega(n+1/2)}{k_BT}}}{Z}\langle m|n\rangle\langle n|m\rangle && (3.12)\\
&= \sum_m \frac{me^{-\frac{\hbar\omega(m+1/2)}{k_BT}}}{Z} \quad \text{See Derivation 2} && (3.13)\\
&= \frac{1}{\exp\frac{\hbar\omega}{k_BT}-1}, && (3.14)
\end{aligned}$$

which is the Bose-Einstein distribution.

Derivation 2: Trick of Sums of Series

Let

$$\tilde{Z}(x) = \sum_{m=0}^{\infty} e^{-mx} = \frac{1}{1-e^{-x}}. \tag{3.15}$$

The trick to calculate the following sums

$$\tilde{Z}_l(x) \equiv \sum_{m=0}^{\infty} m^l e^{-mx}, \tag{3.16}$$

where l is an integer, is from the relation

$$\tilde{Z}_l(x) = (-1)^l \frac{\partial^l \tilde{Z}}{\partial x^l}. \tag{3.17}$$

Substituting Eq. (3.15) into (3.17) and doing the differentiation, you can obtain a closed form of the sum, Eq. (3.16).

Exercise 4: Standard Derivation of $\hat{N}$

Calculate $\sigma\left(\hat{N}\right)$ of an thermal ensemble of temperature T. Use

$$\sigma\left(\hat{N}\right) = \sqrt{\langle\hat{N}^2\rangle - \langle\hat{N}\rangle^2}, \tag{3.18}$$

$$\langle\hat{N}\rangle = \mathrm{Tr}[\rho_{\mathrm{th}}\hat{N}], \tag{3.19}$$

$$\langle\hat{N}^2\rangle = \mathrm{Tr}[\rho_{\mathrm{th}}\hat{N}^2]. \tag{3.20}$$

3.1 Black-Body Radiation

The average energy of one single mode is $\langle\hat{N}\rangle\hbar\omega$. The black-body radiation is defined as the radiation of a large enough thermally equilibrium system. Such a system has the properties

- The system is large enough so that the modes inside are the plane waves $\mathbf{E} = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}$.
- The system is thermally equilibrium so that it has a well-defined temperature T.

For such a system, each $\mathbf{k}$ corresponds to two modes (left/right circular polarizations). The total number of modes M (not photon number) is

$$M = 2\sum_{\mathbf{k}}. \tag{3.21}$$

But, the $\mathbf{k}$ becomes a continuous number when the system is very large. In the continuous limit, it becomes (see Derivation 3)

$$M = \frac{1}{\pi^2}\int_0^\infty k^2 dk. \tag{3.22}$$

This basically gives an infinitely large number since k has no upper bound. We can change the variable of the integral from k to ω by $\omega = ck$

$$M = \int_0^\infty \frac{\omega^2}{\pi^2 c^3} d\omega. \tag{3.23}$$

The M itself is not too meaningful. The number of modes within ω and $\omega + d\omega$ is more meaningful. This is called the density of state $g(\omega)$, given by

$$g(\omega) = \frac{\omega^2}{\pi^2 c^3}. \tag{3.24}$$

With this definition, the total number M

$$M = \int_0^\infty g(\omega) d\omega. \tag{3.25}$$

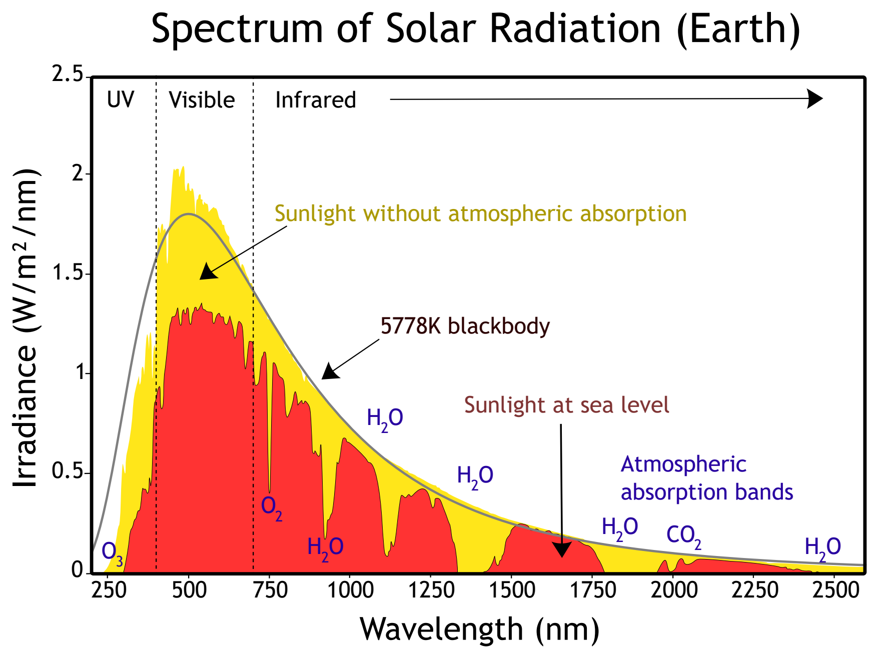


Figure 1: Energy density of a thermal ensemble of photons from Sun and the black-body radiation . (Picture credit: Wikimedia)

The density of state $g(\omega)$ is the number of modes per unit volume within ω and $\omega + d\omega$. The average energy density $U(\omega)$ (energy per unit volume) is then

$$U(\omega) = \langle \hat{N} \rangle g(\omega) \hbar\omega \tag{3.26}$$

$$= \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{\exp \frac{\hbar\omega}{k_B T} - 1}. \tag{3.27}$$

Its classical analog is the Rayleigh-Jeans formula

$$U_{\text{classical}}(\omega) = g(\omega) k_B T = \frac{\omega^2}{\pi^2 c^3} k_B T, \tag{3.28}$$

which leads to the ultraviolet catastrophe of the classical physics, i.e., the energy density diverges as $\omega \to \infty$. The total energy density U_{tot} is

$$U_{\text{tot}} = \int d\omega U(\omega) = \frac{\pi^2 k_B^4 T^4}{15 c^3 \hbar^3}. \tag{3.29}$$

This is the famous Stefan-Boltzmann law, which states that the power radiated by a heated object is proportional to T^4.

Derivation 3: Density of States

A cuboid has the side lengths L_x, L_y and L_z. We assume that the cuboid is large enough so that a plane wave $\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}$ can propagate in any direction. The system should satisfy the periodic boundary conditions so that the allowed wave vector $\mathbf{k} = (k_x, k_y, k_z)$ is

$$k_x = \frac{2\pi l_x}{L_x} \tag{3.30}$$

$$k_y = \frac{2\pi l_y}{L_y} \tag{3.31}$$

$$k_z = \frac{2\pi l_z}{L_z} \tag{3.32}$$

where l_x, l_y and l_z are integers. Note that l_x, l_y and l_z can be negative. The change of the total number m of modes is

$$\Delta m = 2\Delta l_x \Delta l_y \Delta l_z = 2\left(\frac{L_x L_y L_z}{(2\pi)^3}\right)\Delta k_x \Delta k_y \Delta k_z, \tag{3.33}$$

$$\Delta k_x \equiv \frac{2\pi}{L_x}, \tag{3.34}$$

$$\Delta k_y \equiv \frac{2\pi}{L_y}, \tag{3.35}$$

$$\Delta k_z \equiv \frac{2\pi}{L_z}. \tag{3.36}$$

where the factor 2 accounts for the polarizations. In the continuum limit, it becomes

$$\frac{dm}{V} = \left(\frac{1}{4\pi^3}\right) dk_x dk_y dk_z \tag{3.37}$$

$$= \frac{1}{4\pi^3} 4\pi k^2 dk \tag{3.38}$$

$$= \frac{1}{\pi^2}\frac{\omega^2 d\omega}{c^3}, \tag{3.39}$$

$$\Rightarrow g(\omega) \equiv \frac{1}{V}\frac{dm}{d\omega} = \frac{\omega^2}{\pi^2 c^3}. \tag{3.40}$$

Since l_x, l_y and l_z can be negative, it means that k_x, k_y and k_z can be negative. Hence the integral $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_x dk_y dk_z$ can be converted to $\int_0^{\infty} 4\pi k^2 dk$.

Note 5: Paradox: Density of States

As a smart student, you may wonder why do we use the periodic boundary conditions and what if we use the vanishing boundary conditions so that instead of 2π, the the allowed wave vector $\mathbf{k} = (k_x, k_y, k_z)$ becomes

$$k_x = \frac{\pi l_x}{L_x}, \tag{3.41}$$

$$k_y = \frac{\pi l_y}{L_y}, \tag{3.42}$$

$$k_z = \frac{\pi l_z}{L_z}. \tag{3.43}$$

Would this difference lead to a different density of states $g(\omega)$? The answer is no. Of course, the density of states $g(\omega)$ should be the same no matter how one calculate it since there is only one physical truth. The resolution to this paradox is that for the vanishing boundary conditions, the modes are not plane waves but standing waves

$$\mathbf{E} \sim \sin(k_x L_x)\sin(k_y L_y)\sin(k_z L_z). \tag{3.44}$$

Since the modes do not propagate, the wave number are positive $k_x > 0$, $k_y > 0$, and $k_z > 0$ ($l_x > 0$, $l_y > 0$ and $l_z > 0$). Hence, the integrals over k_x, k_y, and k_z start from 0 to ∞. The integral $\int_0^\infty \int_0^\infty \int_0^\infty dk_x dk_y dk_z$ is now converted to $\frac{1}{8}\int_0^\infty 4\pi k^2 dk$. The $\frac{1}{8}$ accounts for that only the shell in the first octant is counted. So overall, you will obtain the same $g(\omega)$. **Actually, the $g(\omega)$ should be the same regardless of the boundaries if the system is large enough.**

4 Quadrature Operators

We have applied the ideas of a harmonic oscillator to quantize fields. The canonical variables of a particle, x and p are numbers. Unlike a particle, the canonical operators of a photon, $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ are vector functions. In other words, to complete determine $\mathbf{E}(\mathbf{r})$, we have to know its value at every position $\mathbf{r}$. In contrast, x does not depend on other coordinates. The similarities of them are the creation and annihilation operators a and $a^\dagger$. It is then useful to define the **dimensionless** operators for photons. We introduce the quadrature operators,

$$X = \frac{a + a^\dagger}{2}, \tag{4.1}$$

$$Y = \frac{a - a^\dagger}{2i}. \tag{4.2}$$

The operator X is the dimensionless position operator, and the operator Y is the dimensionless momentum. They have the relation

$$[X, Y] = \frac{i}{2}. \tag{4.3}$$

Using the generalized uncertainty relation, we obtain

$$\sigma(X)\sigma(Y) \geq \frac{|\langle [X, Y] \rangle|}{2} = \frac{1}{4}. \tag{4.4}$$

The electric field operator of a mode m is rewritten as

$$\mathbf{E}_m(\mathbf{r}) = \mathrm{Re}[\boldsymbol{\mathcal{E}}_m(\mathbf{r})]X - \mathrm{Im}[\boldsymbol{\mathcal{E}}_m(\mathbf{r})]Y. \tag{4.5}$$

In the case of plane waves, the electric field operator of a mode $\{\mathbf{k}, s\}$ is

$$\mathbf{E}_{\mathbf{k},s}(\mathbf{r}) = \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\epsilon_0 V}} \Big\{ \mathrm{Re}[\mathbf{e}_{\mathbf{k},s}] \cos(\mathbf{k}\cdot\mathbf{r})X - \mathrm{Im}[\mathbf{e}^*_{\mathbf{k},s}] \sin(\mathbf{k}\cdot\mathbf{r})Y \Big\}. \tag{4.6}$$

References

[1] P. W. Milonni and M.-L. Shih, *Contemporary Physics*, volume 33, number 5, pages 313-322, 1992

[2] C. Gerry and P. Knight, *Introductory Quantum Optics*, Cambridge University Press, 2005