Beam Splitter and Nonclassical Light

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1 Beam Splitters

A beam splitter is an optical component which is partially transparent. An incident beam on a beam splitter is partially reflected and partially transmitted, and thus split into two beams. Classically, an incident beam with an amplitude A_1 is split into a reflected beam with the amplitude A_1 and a transmitted beam with the amplitude A_2. The amplitudes are related by the coefficients of reflection and transmission,

$$A_2 = rA_1, \tag{1}$$
$$A_3 = tA_1. \tag{2}$$

We can also consider another case where a beam of the amplitude A_0 is incident on the other side of the beam splitter. In this case, the amplitudes are related by

$$A_2 = t'A_0, \tag{3}$$
$$A_3 = r'A_0. \tag{4}$$

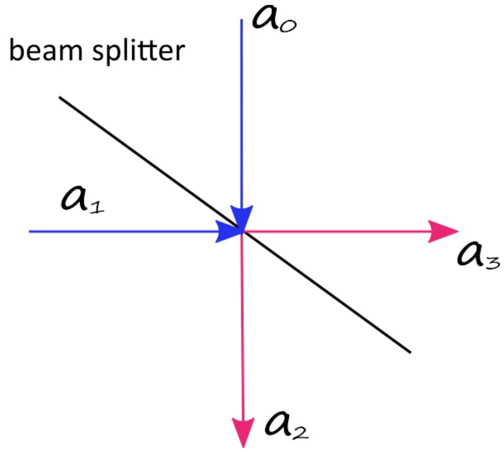

Figure 1: Beam splitter. Quantum descriptions where the amplitudes are replaced by the annihilation operators.

A beam splitter is described by the scattering matrix

$$\begin{pmatrix} A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} \equiv U \begin{pmatrix} A_0 \\ A_1 \end{pmatrix}. \tag{5}$$

Because of energy conservation, the matrix U must be unitary so that

$$|r|^2 + |t|^2 = 1 \tag{6}$$

$$|r'|^2 + |t'|^2 = 1 \tag{7}$$

$$tr^* + t'^* r' = 0. \tag{8}$$

From the last equation, one can find that it is impossible to have $r = r'$ and $t = t'$. This may look inconsistent with the intuition to a symmetric beam splitter where one may speculate $r = r'$ and $t = t'$. The subtlety arises from the phase of each beam. Indeed, for a symmetric beam splitter, only the following conditions are required, $|r|^2 = |r'|^2$ and $|t|^2 = |t'|^2$. In order to satisfy Eq. (8), the phase of each beam cannot be arbitrary. Let the phase between r and t is θ , that is, $\mathrm{Arg}\left[\frac{t}{r}\right] = \theta$. A symmetric beam splitter implies $\mathrm{Arg}\left[\frac{t'}{r'}\right] = \theta$. Eq. (8) becomes

$$|r||t|\left(e^{i\theta} - e^{-i\theta}\right) = 0, \tag{9}$$

which gives $\theta = \pm\frac{\pi}{2}$. A discussion about the phases can be found in the Ref. [1].

The quantum description of a beam splitter is simply to replace amplitudes by annihilation operators. Let the right-going photons have the annihilation operator a_1, and the bottom-going photons have the annihilation operator a_0. A beam splitter is described by the scattering matrix

$$\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = U \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}, \tag{10}$$

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} t'^* & r' \\ r^* & t^* \end{pmatrix} \begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = U^\dagger \begin{pmatrix} a_2 \\ a_3 \end{pmatrix}. \tag{11}$$

The incident beams, a_0 and a_1, are treated as two independent modes with the commutation relations

$$[a_0, a_0^\dagger] = 1, \tag{12}$$

$$[a_1, a_1^\dagger] = 1, \tag{13}$$

$$[a_0, a_1^\dagger] = 0. \tag{14}$$

From Eqs. (12), (13), (14), and (10), one can show that the operators a_2 and a_3 automatically satisfy

$$[a_i, a_j^\dagger] = \delta_{ij}. \tag{15}$$

Physically, these relations mean that after a beam splitter, a beam is split into two independent modes a_2 and a_3.

Below, we are going to discuss what happens to a quantum light after passing a beam splitter. We will consider the cases of a single photon state, N-photon state, and a coherent state. We will see that the Fock states exhibit the quantum natures, where the output states are entangled, while the output state of a coherent state can be factorized.

1.1 Single Photon

The incident state $|0\rangle_0|1\rangle_1$ can be expressed as

$$|0\rangle_0|1\rangle_1 = a_1^\dagger|0\rangle_0|0\rangle_1. \tag{16}$$

From Eq. (11), the creation operators $a_0^\dagger$ and $a_1^\dagger$ are related to $a_2^\dagger$ and $a_3^\dagger$ by

$$\begin{pmatrix} a_0^\dagger \\ a_1^\dagger \end{pmatrix} = U^T \begin{pmatrix} a_2^\dagger \\ a_3^\dagger \end{pmatrix}. \tag{17}$$

After incidence on the beam splitter, the state becomes

$$U^T a_1^\dagger|0\rangle_0|0\rangle_1 = (ra_2^\dagger + ta_3^\dagger)|0\rangle_2|0\rangle_3 \tag{18}$$
$$= r|1\rangle_2|0\rangle_3 + t|0\rangle_2|1\rangle_3, \tag{19}$$

which is an entangled state.

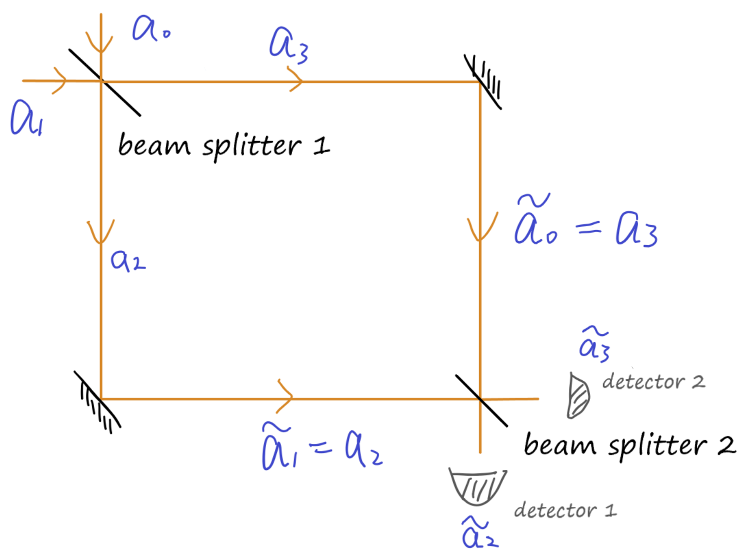

Figure 2: Mach–Zehnder interferometer.

Consider a Mach-Zehnder interferometer with two 50:50 beam splitters of $r = \frac{i}{\sqrt{2}}$ and $t = \frac{1}{\sqrt{2}}$. Let the initial state be $|0\rangle_0|1\rangle_1$. After the first beam splitter, the state becomes

$$\frac{i}{\sqrt{2}}|1\rangle_2|0\rangle_3 + \frac{1}{\sqrt{2}}|0\rangle_2|1\rangle_3. \tag{20}$$

When the state arrives at the second beam splitter, the state becomes

$$\frac{i}{\sqrt{2}}|1\rangle_2|0\rangle_3 + \frac{e^{i\theta}}{\sqrt{2}}|0\rangle_2|1\rangle_3, \tag{21}$$

where θ is a phase shift due to the difference of the two paths.

To deal with the second beam splitter, we first rename the modes. Modes 2 (a_2) and 3 (a_3) in the Equation (21) become the incident beams to the second beam splitter. We rename Mode 2 (a_2) as the new Mode 1 ($\tilde{a}_1$) since it is right-going and Mode 3 (a_3) as the new Mode 0 ($\tilde{a}_0$) since it is bottom-going. For simplicity, we skip the tilde signs below. Now, we can use the same scattering matrix to find the final state after the second beam splitter,

$$\begin{pmatrix} a_2^\dagger \\ a_3^\dagger \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{e^{i\theta}}{\sqrt{2}} a_0^\dagger \\ \frac{i}{\sqrt{2}} a_1^\dagger \end{pmatrix} \tag{22}$$

$$= \begin{pmatrix} \frac{\left(e^{i\theta}-1\right)}{2} a_0^\dagger \\ \frac{i\left(e^{i\theta}+1\right)}{2} a_1^\dagger \end{pmatrix} \tag{23}$$

The probability at D_1 is $\left|\frac{\left(e^{i\theta}-1\right)}{2}\right|^2 = \sin^2\frac{\theta}{2}$. The probability at D_2 is $\left|\frac{i\left(e^{i\theta}+1\right)}{2}\right|^2 = \cos^2\frac{\theta}{2}$. This is a more rigorous description of a single-photon interference.

1.2 N-Photons

Let the initial state be $|0\rangle_0|N\rangle_1 = \frac{(a_1^\dagger)^N}{\sqrt{N!}}|0\rangle$. After a beam splitter, the state becomes

$$\frac{(U a_1^\dagger)^N}{\sqrt{N!}}|0\rangle = \frac{(t a_2^\dagger + r a_3^\dagger)^N}{\sqrt{N!}}|0\rangle. \tag{24}$$

1.3 Coherent States

Let the initial state be $|0\rangle_0|\alpha\rangle_1 = D_1[\alpha]|0\rangle = e^{\alpha a_1^\dagger - \alpha^* a_1}|0\rangle$. After a beam splitter, the state becomes

$$e^{\alpha U a_1^\dagger - \alpha^* U^\dagger a_1}|0\rangle = e^{\alpha t a_2^\dagger - \alpha^* t^* a_2} e^{\alpha r a_3^\dagger - \alpha^* r^* a_3}|0\rangle \tag{25}$$

$$= D_2[t\alpha]D_3[r\alpha]|0\rangle \tag{26}$$

$$= |t\alpha\rangle_2|r\alpha\rangle_3. \tag{27}$$

The input of a coherent state is split into a product of two coherent states. Unlike the single-photon case, this state is not entangled.

Consider a Mach–Zehnder interferometer with two 50:50 beam splitters of $r = \frac{i}{\sqrt{2}}$ and $t = \frac{1}{\sqrt{2}}$. Let the initial state be $|0\rangle_0|\alpha\rangle_1$. After the first beam splitter, the state becomes

$$\left|\frac{i\alpha}{\sqrt{2}}\right\rangle_2 \left|\frac{\alpha}{\sqrt{2}}\right\rangle_3. \tag{28}$$

When the state arrives at the second beam splitter, the state becomes

$$\left|\frac{i\alpha}{\sqrt{2}}\right\rangle_2 \left|\frac{e^{i\theta}\alpha}{\sqrt{2}}\right\rangle_3. \tag{29}$$

where θ is a phase shift due to the difference of the two paths. The final state after the second beam splitter is (see the figure: a_2 is the new a_1 and a_3 is the new a_0)

$$\left|\frac{\left(e^{i\theta}-1\right)}{2}\right\rangle_2 \left|\frac{i\left(e^{i\theta}+1\right)}{2}\right\rangle_3. \tag{30}$$

The intensity at D_1 is $|\alpha|^2\left|\frac{e^{i\theta}-1}{2}\right|^2 = \sin^2\frac{\theta}{2}|\alpha|^2$. The intensity at D_2 is $|\alpha|^2\left|\frac{e^{i\theta}+1}{2}\right|^2 = \cos^2\frac{\theta}{2}|\alpha|^2$. The two output beams are both coherent states. Thus, the phase θ can be obtained by

$$\frac{I_2 - I_1}{|\alpha|^2} = \cos\theta. \tag{31}$$

However, the amplitudes have uncertainty $\sigma(n) = \sqrt{\bar{n}}$. Thus, the phase has the uncertainty $\sigma(\theta) \sim \frac{1}{\sqrt{\bar{n}}}$. In experiments, we would like to use a coherent light (laser) with a well defined phase and a strong intensity such that the uncertainty in phase is small. But a strong-intensity light may lead to more noise such as radiation pressures, thermal noises, and so on. To solve this dilemma, lights with small $\sigma(n)$ is used. These lights are non-classical lights.

2 Quadrature Squeezing

The quadrature operators X and Y, satisfy

$$[X, Y] = \frac{i}{2} \tag{32}$$

$$\Rightarrow \sigma(X)\sigma(Y) \geq \frac{1}{4}. \tag{33}$$

The coherent states satisfy the minimum uncertainty equations,

$$\sigma(X)\sigma(Y) = \frac{1}{4} \tag{34}$$

and

$$\sigma(X) = \sigma(Y) = \frac{1}{2}. \tag{35}$$

which is a circle in the phase space. The conditions of a quadrature squeezing are

$$\sigma(X) < \frac{1}{2} \text{ or } \sigma(Y) < \frac{1}{2} \tag{36}$$

while keeping $\sigma(X)\sigma(Y) = \frac{1}{4}$. Pictorially, a squeezed state is an ellipse in the phase space with a area $\frac{\pi}{16}$. Of course, we can squeeze a state in any direction other than X or Y. We can define the rotated quadrature operator as the following

$$X'(\theta) = \frac{ae^{-i\theta} + a^\dagger e^{i\theta}}{2}, \tag{37}$$

$$Y'(\theta) = \frac{ae^{-i\theta} - a^\dagger e^{i\theta}}{2i}, \tag{38}$$

which represent a coordinate transform of the quadrature operators. Depending on the squeezed axis, we have the following squeezed states. Question: which one has the minimum uncertainty of the photon number?

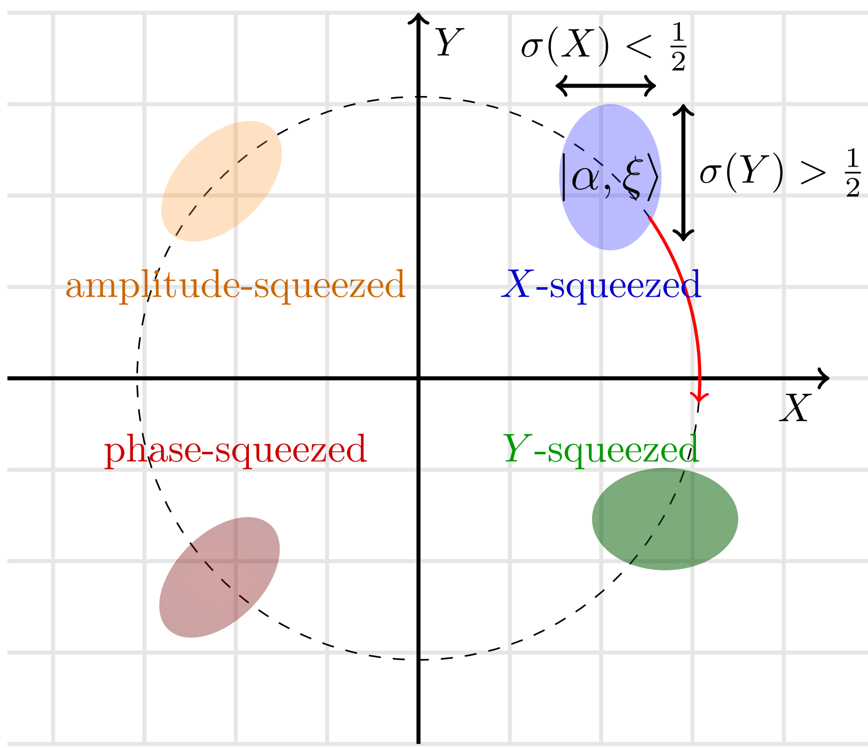

Figure 3: Squeezed States.

2.1 Squeezed Operators

Mathematically, a coherent state is generated by shifting a vacuum state in the phase space. This is done by the displacement operator $D(\alpha)$,

$$|\alpha\rangle = D(\alpha)|0\rangle. \tag{39}$$

We have shown that $D(\alpha)$ is the evolution operator U of a oscillating current source, that is, such a source creates a coherent state.

A squeezed state is generated by a squeeze operator,

$$S(\xi) = \exp\left(\frac{\xi^* a^2 - \xi(a^\dagger)^2}{2}\right), \tag{40}$$

where $\xi = re^{i\theta}$, and r is the squeeze parameter. A squeezed operator is a unitary operator. In principle, a unitary operator correspond a physical process. Observing the quadratic terms of the creation and annihilation operators, it is straightforward to speculate that **the physical processes are nonlinear**. This is because the quadratic terms come from the square of the electric field operators, $\mathbf{E}^2 = \left(\frac{\mathcal{E}a + \mathcal{E}^* a^\dagger}{2}\right)^2$. Squeeze operators have the relations

$$S^\dagger(\xi)S(\xi) = \mathbb{1}, \tag{41}$$

$$S^\dagger(\xi)aS(\xi) = a\cosh r - a^\dagger e^{i\theta}\sinh r, \tag{42}$$

$$S^\dagger(\xi)a^2S(\xi) = \left(a\cosh r - a^\dagger e^{i\theta}\sinh r\right)^2, \tag{43}$$

$$S^\dagger(\xi)a^\dagger S(\xi) = a^\dagger\cosh r - ae^{-i\theta}\sinh r, \tag{44}$$

$$S^\dagger(\xi)(a^\dagger)^2S(\xi) = \left(a^\dagger\cosh r - ae^{-i\theta}\sinh r\right)^2. \tag{45}$$

Let's first consider the squeezing of a vacuum state $S(\xi)|0\rangle$. The uncertainty of the squeezed state is

$$\sigma(X) = \frac{1}{2}\sqrt{\cosh^2 r + \sinh^2 r - 2\sinh r\cosh r\cos\theta}, \tag{46}$$

$$\sigma(Y) = \frac{1}{2}\sqrt{\cosh^2 r + \sinh^2 r + 2\sinh r\cosh r\cos\theta}. \tag{47}$$

When $\theta = 0$,

$$\sigma(X) = \frac{1}{2}e^{-r}, \tag{48}$$

$$\sigma(Y) = \frac{1}{2}e^{r}. \tag{49}$$

The state $S(\xi)|0\rangle$ is called the squeezed vacuum state, which the expectation value of the electric field is zero. We can obtain a more general squeeze state by applying both $D(\alpha)$ and $S(\xi)$ on a vacuum state,

$$|\alpha, \xi\rangle \equiv D(\alpha)S(\xi)|0\rangle. \tag{50}$$

Displacement operators have the relations

$$D^\dagger(\alpha)aD(\alpha) = a + \alpha, \tag{51}$$

$$D^\dagger(\alpha)a^\dagger D(\alpha) = a^\dagger + \alpha^*, \tag{52}$$

which add constants and do not change $\sigma(a)$ and $\sigma(a^\dagger)$. This means that $S(\xi)|0\rangle$ and $D(\alpha)S(\xi)|0\rangle$ have the same shapes in the phase space.

2.2 Number-State Representations

Let $|\xi\rangle = |0,\xi\rangle$ expressed in the number basis,

$$|\xi\rangle = \sum_n C_n |n\rangle., \tag{53}$$

where

$$C_n = \begin{cases} 0, & \text{odd}, \\ \frac{i^n}{\sqrt{\cosh r}} \frac{\sqrt{n!}}{2^{n/2}\left(\frac{n}{2}\right)!} e^{in\theta/2} \tanh^{n/2} r, & \text{even}. \end{cases} \tag{54}$$

For a general squeezed state, $|\alpha,\xi\rangle$, the coefficients are

$$C_n = \exp\left[-\frac{1}{2}|\alpha^2| - \frac{1}{2}(\alpha^*)^2 e^{i\theta} \tanh r\right] \frac{\left(\frac{e^{i\theta}\tanh r}{2}\right)^{n/2}}{\sqrt{n! \cosh r}} H_n\left[\frac{\alpha + \alpha^* e^{i\theta} \tanh r}{\sqrt{2e^{i\theta}\tanh r}}\right]. \tag{55}$$

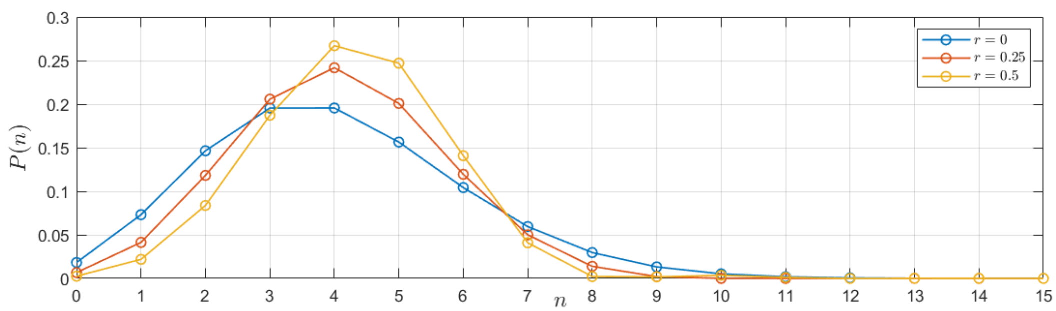

Figure 4: Photon counting of squeezed states. $\alpha = 2$

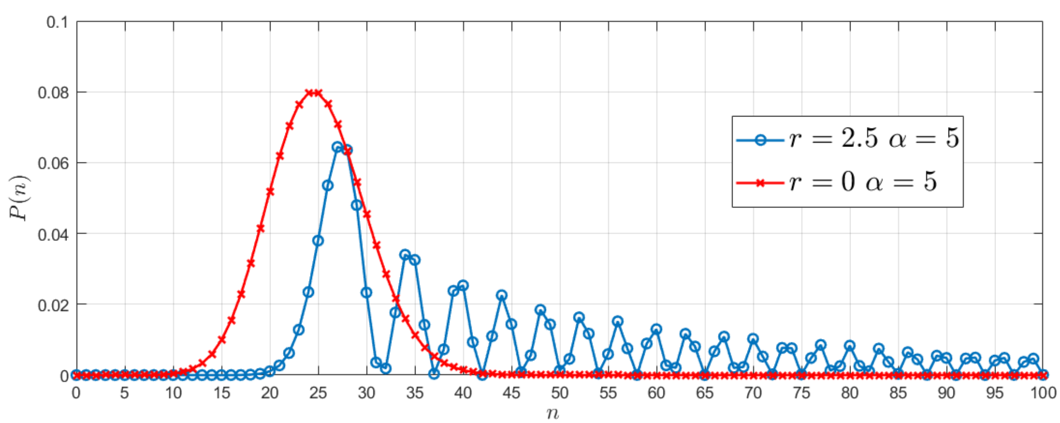

Figure 5: Photon counting of squeezed states. $\alpha = 5$

References

[1] Z. Y. Ou and L. Mandel, Derivation of reciprocity relations for a beam splitter from energy balance, American Journal of Physics 57, 66 (1989)