Light-Matter Interaction: Full Quantum Approaches

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4.1 Jaynes–Cummings Model

The Jaynes–Cummings Model is then obtained as

$$\mathcal{H}_{JC} = \hbar\omega a^\dagger a + \frac{\hbar\omega_{cv}}{2}\sigma_z + \hbar\lambda\left(\sigma_+ a + \sigma_- a^\dagger\right). \tag{1}$$

We have used the Pauli matrices

$$\sigma_z = |E_c\rangle\langle E_c| - |E_v\rangle\langle E_v| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

$$\sigma_+ = |E_c\rangle\langle E_v| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{3}$$

$$\sigma_- = |E_v\rangle\langle E_c| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4}$$

The electron number operator is an identity,

$$N_e = |E_c\rangle\langle E_c| + |E_v\rangle\langle E_v|, \tag{5}$$

and the excitation number operator is

$$N_{ex} = |E_c\rangle\langle E_c| + a^\dagger a. \tag{6}$$

These numbers are conservative since the commutators vanish

$$[\mathcal{H}, N_e] = 0, \tag{7}$$

$$[\mathcal{H}, N_{ex}] = 0, \tag{8}$$

which mean that the total Hamiltonian can be **block-diagonalized**, and in each block, the excitation number and the electron number are the same. The basis kets are

$$|n\rangle \otimes |E_m\rangle \equiv |n\rangle|E_m\rangle \tag{9}$$

where $E_m = E_c$ or E_v and $n = 0,\ 1,\ ,2,\ 3, \ldots$ It seems that if we want to use the number states as the basis, the dimension of the Hamiltonian would be infinite. This is true, but the Hamiltonian can be block-diagonalized. **Because the excitation number is conserved, only the states with the same excitation number are coupled.** Within each block, the excitation number is the same. Eventually, one finds that each block is just a 2 by 2 matrix. This is because the state $|E_c\rangle|n\rangle$ is only coupled to $|E_v\rangle|n+1\rangle$. The problem is then to solve a two-dimensional Hamiltonian since each block is independent.

The Hamiltonian is decomposed as

$$\mathcal{H}_{JC} = \mathcal{H}_N + \mathcal{H}_D \tag{10}$$

$$\mathcal{H}_N = \hbar\omega N_{ex} - \hbar\frac{\omega}{2}N_e, \tag{11}$$

$$\mathcal{H}_D = -\frac{\hbar\Delta}{2}\sigma_z + \hbar\lambda\left(\sigma_+ a + \sigma_- a^\dagger\right). \tag{12}$$

with $\omega = \omega_{cv} + \Delta$. The two Hamiltonians $\mathcal{H}_N$ and $\mathcal{H}_D$ commute with each other,

$$[\mathcal{H}_N, \mathcal{H}_D] = 0, \tag{13}$$

which means the two Hamiltonians are decoupled so

$$e^{-i\frac{\mathcal{H}_N+\mathcal{H}_D}{\hbar}t} = e^{-i\frac{\mathcal{H}_N}{\hbar}t}e^{-i\frac{\mathcal{H}_D}{\hbar}t} = e^{-i\frac{\mathcal{H}_D}{\hbar}t}e^{-i\frac{\mathcal{H}_N}{\hbar}t}. \tag{14}$$

In the basis by Eq. (9), the Hamiltonian $\mathcal{H}_N$ is indeed diagonal, which means that as time increases, $\mathcal{H}_N$ only adds the phase in each basis vector but does not cause the transitions between the basis kets. The physical reason is that the Hamiltonian $\mathcal{H}_N$ describes the conservative numbers so that it is irrelevant to dynamics. Therefore, the dynamics is given by $\mathcal{H}_D$. We can use the interaction picture where $\mathcal{H}_0 = \mathcal{H}_D$ so that the dynamics is given by

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle_I = \mathcal{H}_D|\psi\rangle_I. \tag{15}$$

The ket here is in the interaction picture. Because of being block-diagonalized, the dimension of $|\psi\rangle_I$ is effectively 2.

Example 1: Number State

Let the light in the number state $|n\rangle$. The two basis kets are

$$|n+1\rangle|E_v\rangle \equiv |i\rangle, \tag{16}$$

$$|n\rangle|E_c\rangle \equiv |f\rangle. \tag{17}$$

An arbitrary state in the interaction picture is

$$|\psi(t)\rangle = C_i(t)|i\rangle + C_f(t)|f\rangle. \tag{18}$$

Plugging this state in Eq. (15), we obtain

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} C_f \\ C_i \end{pmatrix} = \begin{pmatrix} -\frac{\hbar\Delta}{2} & \sqrt{n+1}\hbar\lambda \\ \sqrt{n+1}\hbar\lambda & \frac{\hbar\Delta}{2} \end{pmatrix}\begin{pmatrix} C_f \\ C_i \end{pmatrix}. \tag{19}$$

The eigenfrequencies are

$$\omega_{\pm} = \pm\sqrt{\frac{\Delta^2}{4} + (n+1)\lambda^2}. \tag{20}$$

and the eigenvectors (using the Bloch sphere representation) are

$$|\omega_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} e^{-i\omega_+ t} \tag{21}$$

$$|\omega_-\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix} e^{-i\omega_- t} \tag{22}$$

with

$$\theta = -\tan^{-1}\left(\frac{2\sqrt{n+1}\lambda}{\Delta}\right). \tag{23}$$

If the initial state is $C_i = 1$ and $C_f = 0$, the solution becomes

$$|\psi\rangle = \sin\frac{\theta}{2}|\omega_+\rangle - \cos\frac{\theta}{2}|\omega_-\rangle, \tag{24}$$

$$C_i(t) = \cos\omega_+ t + i\cos\theta\sin\omega_+ t, \tag{25}$$

$$C_f(t) = -i\sin\theta\sin\omega_+ t. \tag{26}$$

The population of the excited state $n_e = |C_f(t)|^2$ is

$$n_e = \sin^2\theta\sin^2\omega_+ t, \tag{27}$$

$$= \sin^2\theta\sin^2\sqrt{\frac{\Delta^2}{4} + (n+1)\lambda^2}t. \tag{28}$$

This is the Rabi oscillation between the states $|E_v\rangle|n+1\rangle$ and $|E_c\rangle|n\rangle$. Only when the detuning is zeros, we have $\sin\theta = 1$ and the maximum excitation. The Rabi frequency is

$$\omega_+ = \sqrt{\frac{\Delta^2}{4} + (n+1)\lambda^2}. \tag{29}$$

The Rabi frequency does depend on the number of the photons. One novel case is $n = 0$ where the frequency is not zero but

$$\omega_+(n=0) = \sqrt{\frac{\Delta^2}{4} + \lambda^2}. \tag{30}$$

This means that there exists the Rabi oscillation even when there is no photon.[a] This is called the "vacuum Rabi oscillations".

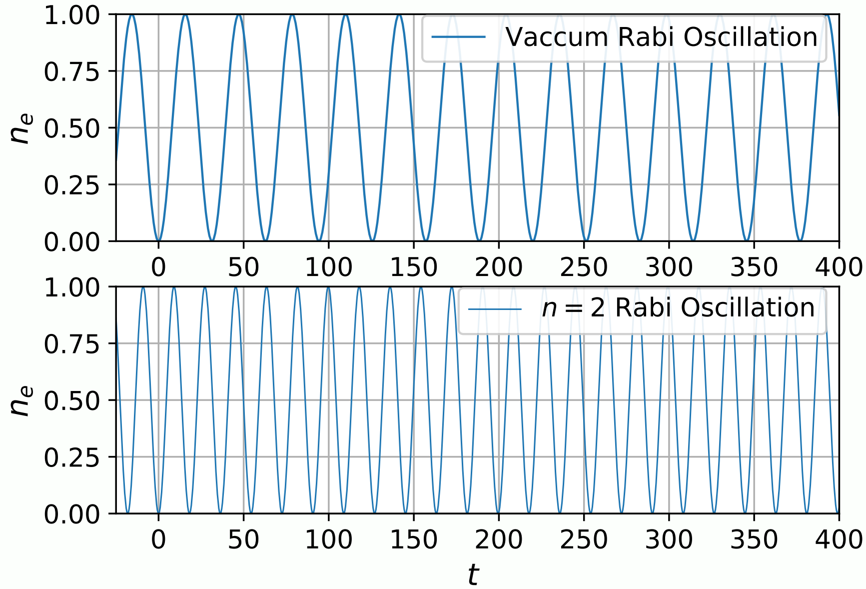

Figure 1: Rabi oscillations of the JC models for $n = 0$ and $n = 2$. The other parameters are $\Delta = 0$ and $\lambda = 0.1$

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[a]Though, the vacuum energy is nonzero!

4.2 JC models with a Coherent State

Let us consider a more general situation where the photon state is

$$|\text{field}\rangle = \sum_{n=0}^{\infty} C_n |n\rangle, \tag{31}$$

and the two level system is

$$|\text{TLS}\rangle = C_c |E_c\rangle + C_v |E_v\rangle. \tag{32}$$

The total state is

$$|\psi\rangle = |\text{field}\rangle \otimes |\text{TLS}\rangle. \tag{33}$$

The solution is then (when $\Delta = 0$)

$$|\psi\rangle = \sum_n [C_c C_n \cos(\omega_{n+1} t) - i C_v C_{n+1} \sin(\omega_{n+1} t)] |n\rangle |E_c\rangle \tag{34}$$

$$+ \sum_n [C_v C_{n+1} \cos(\omega_{n+1} t) - i C_c C_n \sin(\omega_{n+1} t)] |n+1\rangle |E_v\rangle, \tag{35}$$

where

$$\omega_n = \omega_+(n). \tag{36}$$

Let the initial state be $C_c = 0$ and $C_v = 1$. The population of the excited state is

$$n_e = |C_c(t)|^2 = \sum_n |C_{n+1}|^2 \sin^2 \omega_{n+1} t \tag{37}$$

$$= \sum_n |C_{n+1}|^2 \left(\frac{1 - \cos 2\omega_{n+1} t}{2} \right) \tag{38}$$

$$= \frac{1}{2} - \sum_n |C_{n+1}|^2 \left(\frac{\cos 2\omega_{n+1} t}{2} \right). \tag{39}$$

In terms of n, we obtain

$$n_e = \frac{1}{2} - \sum_n |C_{n+1}|^2 \left(\frac{\cos 2\lambda \sqrt{n+1}\, t}{2} \right). \tag{40}$$

Figure 2 shows the populations in the cases of coherent states. Even with a coherent state, the population is not a simple harmonic oscillation as in the classical case. There are two new properties. First, the oscillation lasts for a time τ_c (the duration of the wave packet.) and **collapses**. It is shown that the time τ_c is in the limit $n \to \infty$,

$$\tau_c \simeq \frac{\sqrt{2}}{\lambda}. \tag{41}$$

After a rephasing time $\tau_{\rm rp}$, the oscillation comes back. This is called the **revival**. The time $\tau_{\rm rp}$ is in the limit $n \to \infty$,

$$\tau_{\rm rp} \simeq \frac{4\pi|\alpha|}{\lambda}. \tag{42}$$

Two properties of the JC model are

- Collapsing
- Revival

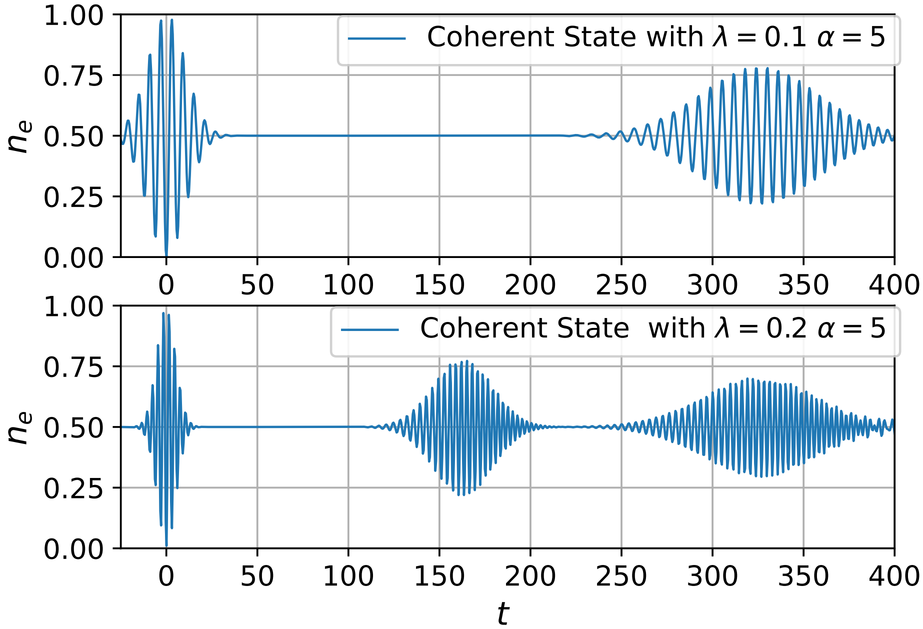

Figure 2: Rabi oscillations of the JC models for a coherent state. Collapsing and revival appear.

4.3 Dressed States

We focused on the dynamics of the JC model. Now, we discuss the eigenstates of the JC model. First, the photon energy in the vaccuum is $E = n\hbar\omega$.[1] In a cavity, photons are coupled with the TLS. As a result, the photon energies are shifted. We can think that the combination of photons and the TLS leads to a new state called the "dressed state", or in the context of condensed matter physics, "polaritons". We start with the full Hamiltonian,

$$\mathcal{H} = \hbar\omega a^\dagger a - \hbar\Delta\sigma_z + \hbar\lambda(\sigma_- a^\dagger + \sigma_+ a). \tag{43}$$

Consider the subspace spanned by Eqs. (16) and (17). The eigenvalues are

$$E_{1n} = n\hbar\omega + \omega_n, \tag{44}$$
$$E_{2n} = n\hbar\omega - \omega_n, \tag{45}$$

where $\omega_n = \sqrt{\frac{\Delta^2}{4} + (n+1)\lambda^2}$ and the eigenvectors (using the Bloch sphere representation) are

$$|1n\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} e^{-i\omega_+ t} \tag{46}$$

$$|2n\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix} e^{-i\omega_- t} \tag{47}$$

[1]We drop $1/2\hbar\omega$.

with

$$\theta = -\tan^{-1}\left(\frac{2\sqrt{n+1}\lambda}{\Delta}\right). \tag{48}$$

The dressed photons are the eigenstates of the total system. Compared to photons in vacuum, their frequencies shift and become non-degenerate. The splitting of dressed states is the origin of the Mollow triplet emissions.

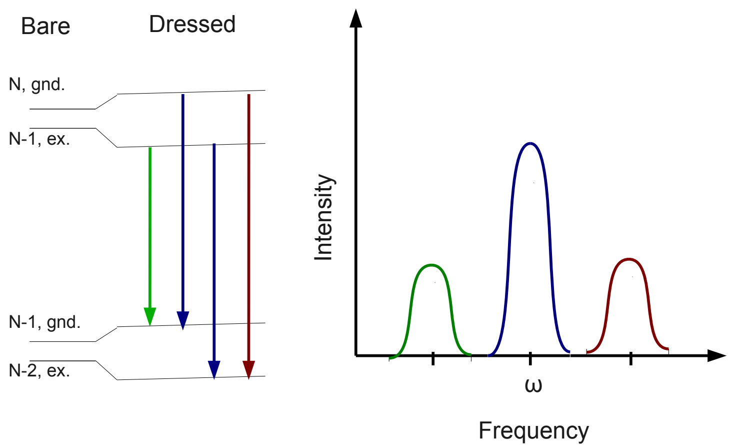

Figure 3: Mollow triplet emissions.

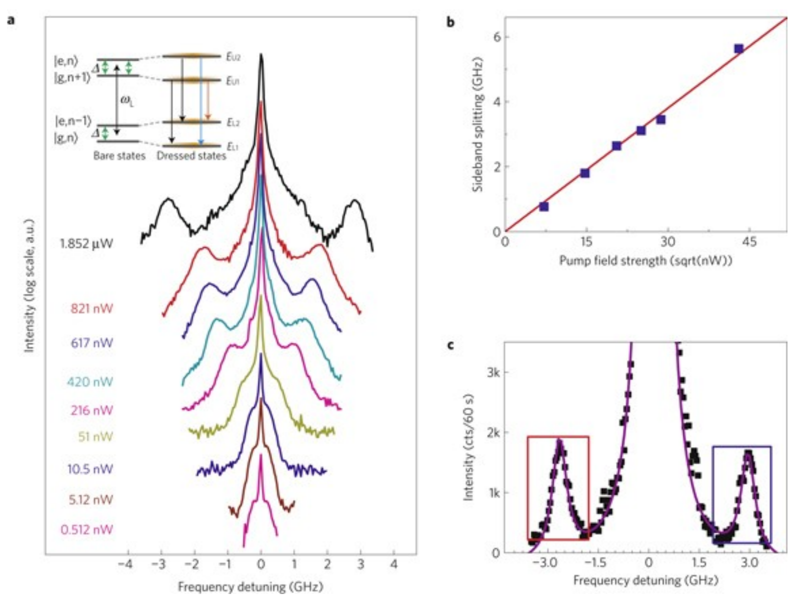

Figure 4: Experimental observation of the Mollow triplet emissions. From Nature Physics 5, 198–202(2009)