Light-Matter Interaction

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Light-matter interactions occur when charged particles accelerate in time-dependent electric field. An accelerating charge particle generates light, and conversely electric fields cause forces on charges particle. In most scenarios, magnetic field does not directly interact with matter since it is easier to have charges, electric dipoles than magnetic dipoles.

Time-dependent charges can be described by a charge density $\rho(\mathbf{r},t)$. It is more often to use dipoles and currents to describe light-matter interaction. Polarization $\mathbf{P}$ (dipole) and currents density $\mathbf{J}$ have the relations

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0, \tag{1}$$

$$\mathbf{J} = \frac{\partial \mathbf{P}}{\partial t}. \tag{2}$$

1 Hamiltonian

1.1 Interaction Hamiltonian

According to classical mechanics, a charge particle has the Hamiltonian (SI units)

$$\mathcal{H} = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} + q\Phi(\mathbf{r},t), \tag{3}$$

where q is the charge of the particle not the position. $\Phi(\mathbf{r},t)$ is the electric potential. In the case of an electron, $q = -e$, we have

$$\mathcal{H} = \frac{(\mathbf{p} + e\mathbf{A})^2}{2m} - e\Phi(\mathbf{r},t). \tag{4}$$

We can decompose it into $\mathcal{H}_0$ and $\mathcal{H}_I$,

$$\mathcal{H}_0 = \frac{p^2}{2m}, \tag{5}$$

$$\mathcal{H}_I = \frac{e(\mathbf{p}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{p})}{2m} + \frac{e^2A^2}{2m} - e\Phi. \tag{6}$$

Typically, the term $\frac{e^2A^2}{2m}$ is dropped since the momentum of field $e\mathbf{A}$ is usually small than the electron's momentum $\mathbf{p}$ [1]. Since the momentum $\mathbf{p}$ is a differential operator, $\mathbf{p}\cdot\mathbf{A}$ is not equal

[1] Well, this is a sloppy argument. In electromagnetism, the higher-order terms of the vector potential$\mathbf{A}$ are relativistic. In this viewpoint, the term $\frac{e^2A^2}{2m}$ is proportional to $\frac{v^2}{c^2}$.

to $\mathbf{A}\cdot\mathbf{p}$. The vector potential $\mathbf{A}$ and Coulomb's potential Φ are not unique. The Maxwell's equations are invariant under the gauge transformations

$$\mathbf{A}' = \mathbf{A} + \nabla\lambda(\mathbf{r}, t), \tag{7}$$

$$\Phi' = \Phi - \frac{\partial\lambda(\mathbf{r}, t)}{\partial t}. \tag{8}$$

The fields are given by

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{9}$$

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}. \tag{10}$$

The Gauge $\nabla\cdot\mathbf{A} = 0$ is frequently used in quantum optics. In this gauge, the interaction Hamiltonian (dropping $\frac{e^2A^2}{2m}$) becomes

$$\mathcal{H}_I = \frac{e(\mathbf{A}\cdot\mathbf{p})}{m} - e\Phi. \tag{11}$$

If $\Phi = 0$ is chosen [2], the interaction Hamiltonian becomes

$$\mathcal{H}_I = \frac{e(\mathbf{A}\cdot\mathbf{p})}{m} \tag{12}$$

$$= -\int dv\mathbf{A}\cdot\mathbf{J} \tag{13}$$

where we use $\int dv\mathbf{J} = \frac{-e\mathbf{p}}{m}$. Another choice is the Göppert-Mayer gauge,

$$\lambda = -(\mathbf{r} - \mathbf{r}_0)\cdot\mathbf{A}(\mathbf{r}_0). \tag{14}$$

Using this gauge and Eq. (8), we have

$$\mathbf{A}' = \mathbf{A}(\mathbf{r}) - \mathbf{A}(\mathbf{r}_0), \tag{15}$$

$$e\Phi' = -e(\mathbf{r} - \mathbf{r}_0)\cdot\mathbf{E}(\mathbf{r}_0) \equiv -\mathbf{d}\cdot\mathbf{E}, \tag{16}$$

where $\mathbf{d} = -e(\mathbf{r} - \mathbf{r}_0)$ is the dipole operator since $\mathbf{r}$ is the position operator. The so-called dipole approximation is when $\mathbf{A}(\mathbf{r})$ is almost a constant, i.e., $\mathbf{A}(\mathbf{r}) \simeq \mathbf{A}(\mathbf{r}_0)$. In this approximation, the new vector potential $\mathbf{A}'$ vanishes. This approximation is valid if the field changes gradually over the range of the charge distributions. For example, the charge distribution of an atom is about 0.1 nm, and the electric filed of visible lights is almost a constant over the atom since the wavelengths range from 400 to 700 nm. The interaction Hamiltonian becomes

$$\mathcal{H}_I = -\mathbf{E}\cdot\mathbf{d} \tag{17}$$

[2] In the region without charges $\nabla\cdot\mathbf{E} = 0$, we can define $\mathbf{E} = -\nabla\Phi$. Using the gauge transformation $\lambda = \int\Phi dt$, we can eliminate Φ and make $\nabla\cdot\mathbf{A} = 0$.

1.2 Total Hamiltonian

The total Hamiltonian of the light-matter is

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I + \mathcal{H}_F. \tag{18}$$

where

$$\mathcal{H}_F = \sum_m \int dv \left(\frac{\epsilon(\mathbf{r}) E_m^2(\mathbf{r})}{2} + \frac{B_m^2(\mathbf{r})}{2\mu(\mathbf{r})} \right) \tag{19}$$

$$= \sum_m \hbar\omega_m \left(a_m^\dagger a_m + \frac{1}{2} \right). \tag{20}$$

The Hamiltonian of matter $\mathcal{H}_0$ is not necessary of the form of a free particle. In general, $\mathcal{H}_0$ describes a N-level system,

$$\mathcal{H}_0 = \sum_n E_n |E_n\rangle\langle E_n|. \tag{21}$$

The simplest case is a two level system (TLS)

$$\mathcal{H}_{TLS} = \begin{pmatrix} E_c & 0 \\ 0 & E_v \end{pmatrix}. \tag{22}$$

The interaction Hamiltonian for a a two level system is

$$\mathcal{H}_I = \begin{pmatrix} \langle E_c| - \mathbf{E}\cdot\mathbf{d}|E_c\rangle & \langle E_c| - \mathbf{E}\cdot\mathbf{d}|E_v\rangle \\ \langle E_v| - \mathbf{E}\cdot\mathbf{d}|E_c\rangle & \langle E_v| - \mathbf{E}\cdot\mathbf{d}|E_v\rangle \end{pmatrix} \tag{23}$$

$$= -\mathbf{E}\cdot\begin{pmatrix} \mathbf{d}_{cc} & \mathbf{d}_{cv} \\ \mathbf{d}_{vc} & \mathbf{d}_{vv} \end{pmatrix}, \tag{24}$$

where the dipole matrix element is $\mathbf{d}_{nn'} = \langle E_n|\mathbf{d}|E_{n'}\rangle$. In many cases, the diagonal elements of dipole matrices vanishes since the charge densities of the eigenfunctions are typical symmetric.

2 Classical Fields and Quantum Matter

We consider that the matter is described by a N-level system and treat the electric field $\mathbf{E}(\mathbf{r}, t)$ as a number. The Hamiltonian is

$$\mathcal{H} = \sum_n E_n |E_n\rangle\langle E_n| - \mathbf{E}\cdot\mathbf{d}. \tag{25}$$

In the case of a TLS system, the Hamiltonian is

$$\mathcal{H} = \begin{pmatrix} E_c & 0 \\ 0 & E_v \end{pmatrix} - \mathbf{E}\cdot\begin{pmatrix} 0 & \mathbf{d}_{cv} \\ \mathbf{d}_{vc} & 0 \end{pmatrix}, \tag{26}$$

where we assume the diagonal elements of the dipole matrix are zeros. To solve the dynamics, we start with the interaction picture where state is

$$|\psi\rangle = C_c(t)e^{-i\omega_c t}|E_c\rangle + C_v(t)e^{-i\omega_v t}|E_v\rangle. \tag{27}$$

It is clear that without an external field $\mathbf{E}$, the coefficients $C_c(t)$ and $C_v(t)$ are constant in time. Plugging Eq. (27) in the Schrödinger equation, we obtain

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} C_c \\ C_v \end{pmatrix} = -\mathbf{E}\cdot\begin{pmatrix} 0 & \mathbf{d}_{cv}e^{i(\omega_c-\omega_v)t} \\ \mathbf{d}_{vc}e^{i(\omega_v-\omega_c)t} & 0 \end{pmatrix}\begin{pmatrix} C_c \\ C_v \end{pmatrix}. \tag{28}$$

The dipole matrix elements in the interaction picture oscillate rapidly in time. The electric field $\mathbf{E} = \boldsymbol{\mathcal{E}}_\omega e^{-i\omega t} + \boldsymbol{\mathcal{E}}^*_\omega e^{i\omega t}$ needs to have a frequency $\omega \simeq (\omega_c - \omega_v)$ in order to create transition. We write

$$\omega = \omega_{cv} + \Delta, \tag{29}$$

where $\omega_{cv} = \omega_c - \omega_v$ and Δ is the detuning.

2.1 Rabi Model

Let the external field $\mathbf{E} = \mathbf{E}_0\cos\omega t = \mathbf{E}_0\left(\frac{e^{-i\omega t}+e^{i\omega t}}{2}\right)$. The equation of the coefficients is

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} C_c \\ C_v \end{pmatrix} = \begin{pmatrix} 0 & \frac{V_0}{2}\left[e^{-i\Delta t} + e^{i(2\omega_{cv}+\Delta)t}\right] \\ \frac{V_0^*}{2}\left[e^{i\Delta t} + e^{-i(2\omega_{cv}+\Delta)t}\right] & 0 \end{pmatrix}\begin{pmatrix} C_c \\ C_v \end{pmatrix}. \tag{30}$$

where

$$V_0 = -\mathbf{E}_0\cdot\mathbf{d}_{cv}. \tag{31}$$

The equation needs to be solved numerically. The rotating-wave-approximation (RWA), where the high frequency terms are dropped is often used. Under the RWA, the equation reads

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} C_c \\ C_v \end{pmatrix} = \begin{pmatrix} 0 & \frac{V_0}{2}e^{-i\Delta t} \\ \frac{V_0^*}{2}e^{i\Delta t} & 0 \end{pmatrix}\begin{pmatrix} C_c \\ C_v \end{pmatrix}. \tag{32}$$

Eliminating the variable C_v, we obtain the second-order differential equation

$$\ddot{C}_c + i\Delta\dot{C}_c + \frac{|V_0|^2}{4\hbar^2}C_c = 0. \tag{33}$$

The general solution is

$$C_c(t) = A_+e^{i\lambda_+ t} + A_-e^{i\lambda_- t} \tag{34}$$

with

$$\lambda_{\pm} = \Delta \pm \sqrt{\Delta^2 + \frac{|V_0|^2}{\hbar^2}} \equiv \Delta \pm \Omega_R. \tag{35}$$

The Rabi frequency $\Omega_R = \sqrt{\Delta^2 + \frac{|V_0|^2}{\hbar^2}}$. If initially $C_v(0) = 1$ and $C_c(0) = 1$, the solution is

$$C_c = e^{i\frac{\Delta t}{2}} \frac{iV_0}{\hbar\Omega_R} \sin\frac{\Omega_R t}{2}, \tag{36}$$

$$C_v = e^{i\frac{\Delta t}{2}} \left[\cos\frac{\Omega_R t}{2} - i\frac{\Delta}{\Omega_R}\sin\frac{\Omega_R t}{2}\right]. \tag{37}$$

It can be checked that $|C_c|^2 + |C_v|^2 = 1$. The population of the excited state is

$$P_c(t) = |C_c(t)|^2 = \frac{|V_0|^2 \sin^2\frac{\Omega_R t}{2}}{\hbar^2\Omega_R^2} \tag{38}$$

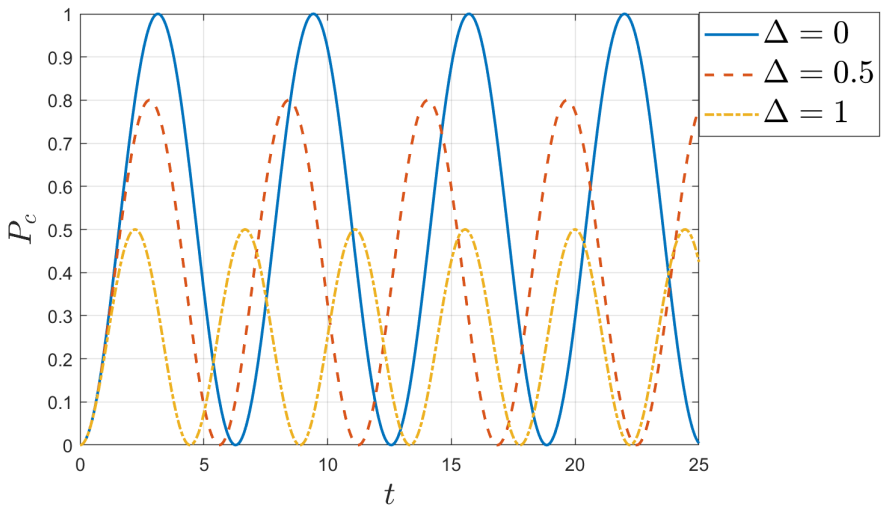

Figure 1: Population of the excited state as a function of time with $\frac{V_0}{\hbar} = 1$.

2.2 Fermi's Golden Rule

If the external field is small, we can obtain from Eq. (36) [3]

[3] the formal method to obtain this result is the the perturbation method (for example, see Chapter 5 of Ref. [1])

$$P_c(t) = |C_c|^2 = \frac{|V_0|^2 \sin^2 \frac{\Delta t}{2}}{\hbar^2 \Delta^2}. \tag{39}$$

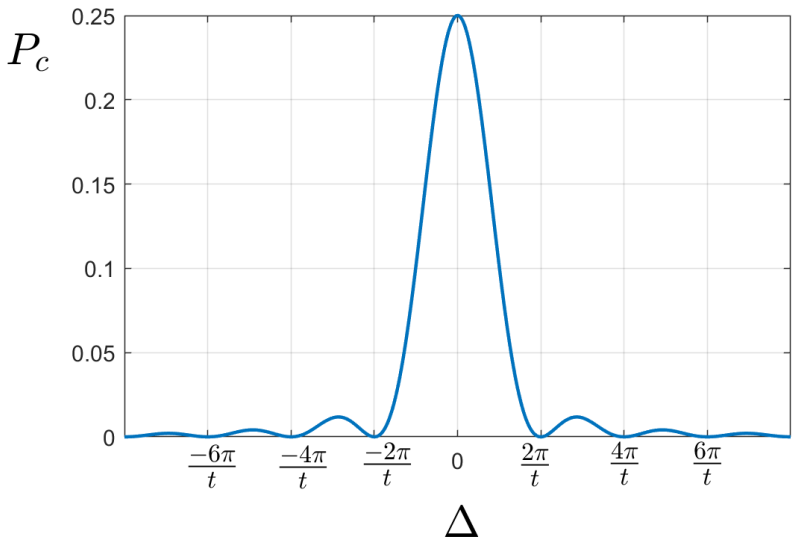

Figure 2: The transition probability $P_c(t)$ at a momentum t. When t is large, the function is approximately a delta function.

When t is large, the fraction is approximately a delta function

$$\frac{\sin^2 \frac{\Delta t}{2}}{\Delta^2} \simeq \frac{\pi t}{2} \delta(\Delta). \tag{40}$$

The transition rate $W_{v\to c}$ is

$$W_{v\to c} = \frac{P_c(t)}{t} = \frac{\pi}{2} \frac{|V_0|^2}{\hbar^2} \delta(\omega - \omega_{cv}) \tag{41}$$

$$= \frac{\pi}{2} \frac{|\mathbf{E}_0 \cdot \mathbf{d}_{cv}|^2}{\hbar^2} \delta(\omega - \omega_{cv}) \tag{42}$$

$$= \frac{\pi}{2} \frac{|\langle c|\mathbf{H}_I|v\rangle|^2}{\hbar^2} \delta(\omega - \omega_{cv}), \tag{43}$$

which is the famous Fermi's Golden rule.

3 Classical Matter and Quantum Fields

Currents and charges are treated as classical numbers. Time-dependent charges and currents are not independent variables. They are related by the continuity equation. This assumption is

adequate when currents come form a lot of electrons and the quantum fluctuations are ignored. The typical problem is how a current source $\mathbf{I}(\mathbf{r},t)$ interacts with photons. The current is a control and macroscopic parameter which can be treated classically as a number. Thus, currents are given functions, and the problem is to solve filed Hamiltonian.

$$\mathcal{H} = \mathcal{H}_F + \mathcal{H}_I \tag{44}$$

$$= \sum_m \hbar\omega_m a_m^\dagger a_m - \sum_m \mathbf{E}_m \cdot \mathbf{d} \tag{45}$$

$$= \sum_m \hbar\omega_m a_m^\dagger a_m - \sum_m \left(\frac{\boldsymbol{\mathcal{E}}_m a + \boldsymbol{\mathcal{E}}_m^* a^\dagger}{2}\right) \cdot \mathbf{d}, \tag{46}$$

The above interaction Hamiltonian has the dipole instead of a current. Dynamically, dipoles and currents are related. Let the current be $\mathbf{I}(\mathbf{r},t) = \mathbf{I}(\mathbf{r})_0 e^{-i\omega t}$. The current is related to the current density $\mathbf{J}$ by

$$\mathbf{J}(\mathbf{r},t) = \frac{\mathbf{I}(\mathbf{r},t)}{d a_\perp}. \tag{47}$$

From this relation, we can find the current density $\mathbf{J}(\mathbf{r},t) = \mathbf{J}_0(\mathbf{r})e^{-i\omega t}$. Now we can use the interaction Hamiltonian in terms of $\mathbf{J}$ and $\mathbf{A}$. Considering a single mode and $\omega_m = \omega$, the Hamiltonian becomes

$$\mathcal{H} = \hbar\omega a^\dagger a - \int dv \mathbf{A} \cdot \mathbf{J}. \tag{48}$$

The relation of the electric filed operator and the vector potential operator is given by

$$\boldsymbol{\mathcal{E}} = -\frac{\partial}{\partial t}\boldsymbol{\mathcal{A}}, \tag{49}$$

$$\mathbf{A} = \left(\frac{\boldsymbol{\mathcal{E}} a - \boldsymbol{\mathcal{E}}^* a^\dagger}{2i\omega}\right). \tag{50}$$

3.1 Generation of Coherent States

We are going to show a coherent state $|\alpha\rangle$ can be generated by a harmonic oscillating current density

$$\mathbf{J} = \frac{\mathbf{J}_0(\mathbf{r})e^{-i\omega t} + \mathbf{J}_0^*(\mathbf{r})e^{i\omega t}}{2} \tag{51}$$

This current density oscillating with the frequency ω can excite photons of the same frequency. The total Hamiltonian is

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I, \tag{52}$$

with the photon Hamiltonian $\mathcal{H}_0 = \hbar\omega a^\dagger a$ and $\mathcal{H}_I = -\int dv \mathbf{A}\cdot\mathbf{J}$. Using Eq. (50) and the RWA, the interaction Hamiltonian becomes

$$\mathcal{H}_I = \left(V_0 a + V_0^* a^\dagger\right), \tag{53}$$

where

$$V_0(t) = \frac{e^{i\omega t}\int dv \boldsymbol{\mathcal{E}}_\omega^*(\mathbf{r})\cdot\mathbf{J}_0(\mathbf{r})}{4i\omega}. \tag{54}$$

Now the term $V_0(t)$ is time-dependent. We can use the interaction picture to remove the time-dependence. In the interaction picture[4], the interaction Hamiltonian becomes[5]

$$\tilde{\mathcal{H}}_I = \left(V_I a + V_I^* a^\dagger\right), \tag{55}$$

where the interaction potential becomes time-indepdenent and reads

$$V_I = \frac{\int dv \boldsymbol{\mathcal{E}}_\omega^*(\mathbf{r})\cdot\mathbf{J}_0(\mathbf{r})}{4i\omega}. \tag{56}$$

The evolution of a state is given by

$$|\psi(t)\rangle_I = \hat{\mathcal{T}}[e^{-i\int\frac{\tilde{\mathcal{H}}_I(t)}{\hbar}dt}]|\psi(0)\rangle_I \tag{57}$$

where $\hat{\mathcal{T}}[]$ denotes the time-ordering[6]. In this case, the interaction Hamiltonian in the interaction picture is time-independent,

$$|\psi(t)\rangle_I = e^{-i\frac{\tilde{\mathcal{H}}_I(t)}{\hbar}t}|\psi(0)\rangle_I \tag{58}$$

$$= e^{\alpha^* a - \alpha a^\dagger}|\psi(0)\rangle_I, \tag{59}$$

where

$$\alpha = i\frac{V_I^*}{\hbar}t. \tag{60}$$

Equation (59) is indeed the displacement operator. If the initial state is the ground state $|0\rangle$, the final state is a coherent state,

$$|\psi(t)\rangle_I = e^{\alpha^* a - \alpha a^\dagger}|0\rangle \tag{61}$$

$$= |\alpha\rangle. \tag{62}$$

One interesting observation is that $|\alpha| \sim t$ and the photon number $n \sim t^2$ grows quadratically.

[4]Rotating with the $\mathcal{H}_0$.

[5]To avoid confusion, we use $\tilde{\mathcal{H}}_I$ to denote the interaction Hamiltonian in the interaction picture.

[6]Time-ordering is necessary if H_I is time-dependent and $[H_I(t_1), H_I(t_2)] \neq 0$

4 Fully Quantum Approach

When both electrons and fields are quantized, the Hamiltonian includes the three parts: photons, electrons, and the interactions. The Hamiltonian is

$$\mathcal{H} = \mathcal{H}_F + \mathcal{H}_e + \mathcal{H}_I \tag{63}$$

$$= \sum_m \hbar\omega_m a_m^\dagger a_m + \sum_n E_n |E_n\rangle\langle E_n| - \mathbf{E}\cdot\mathbf{d}. \tag{64}$$

It should be noted that both the field $\mathbf{E}$ and the dipole $\mathbf{d}$ are operators. The electric field operator is

$$\mathbf{E} = \sum_m \frac{\boldsymbol{\mathcal{E}}_m a_m + \boldsymbol{\mathcal{E}}_m^* a_m^\dagger}{2}, \tag{65}$$

and the dipole matrix operator in the energy basis is

$$\begin{pmatrix} \mathbf{d}_{11} & \mathbf{d}_{12} & \cdots \\ \mathbf{d}_{21} & \mathbf{d}_{22} & \\ \vdots & & \ddots \end{pmatrix}, \tag{66}$$

with $\mathbf{d}_{nn'} = \langle E_n|\mathbf{d}|E_{n'}\rangle$ and $\mathbf{d} = q\mathbf{r} = -e\mathbf{r}$.

The Hilbert space of the Hamiltonian includes both the photon and electron parts. The total space is indeed the tensor direct product of each space,

$$|\psi\rangle = |\text{photon}\rangle \otimes |\text{electron}\rangle. \tag{67}$$

The dimension of the total space is the product of the dimension of each the space. In this definition, the photonic operators such as a and $a^\dagger$ will only be applied on the photonic ket $|\text{photon}\rangle$, and the electronic operators such as $\mathbf{d}$ will only be applied on the electronic ket $|\text{electron}\rangle$.

$$\langle\psi|\mathcal{H}_F|\psi\rangle = \langle\text{photon}|\mathcal{H}_F|\text{photon}\rangle \otimes \langle\text{electron}|\text{electron}\rangle = \langle\text{photon}|\mathcal{H}_F|\text{photon}\rangle \otimes \mathbb{1}_e, \tag{68}$$

$$\langle\psi|\mathcal{H}_e|\psi\rangle = \langle\text{photon}|\text{photon}\rangle \otimes \langle\text{electron}|\mathcal{H}_|\text{electron}\rangle = \mathbb{1}_F \otimes \langle\text{photon}|\mathcal{H}_e|\text{photon}\rangle, \tag{69}$$

$$\langle\psi|\mathbf{E}\cdot\mathbf{d}|\psi\rangle = \langle\text{photon}|\mathbf{E}|\text{photon}\rangle \cdot \langle\text{electron}|\mathbf{d}|\text{electron}\rangle. \tag{70}$$

For example, we can write the photonic ket in the number basis and the electron ket in the energy basis,

$$|\text{photon}\rangle = \sum_n C_n |n\rangle, \tag{71}$$

$$|\text{electron}\rangle = \sum_m D_m |E_m\rangle. \tag{72}$$

Now all the possible states of the total space can be written as

$$|\psi\rangle = \left(\sum_n C_n|n\rangle\right) \otimes \left(\sum_m D_m|E_m\rangle\right). \tag{73}$$

In principle, the dimension f the total space is infinite since the dimension of the number state is infinite. In practical computation, we will truncate the photon number so that the maximum number is finite, say n_m. The photon basis vectors now include $|0\rangle$, $|1\rangle$,... , $|n_m\rangle$, so the dimension of the photonic part is m. If now we consider a two-level system of electrons, the dimension of the total space is $m\times 2$. All the basis vectors of the total space are $|0\rangle|E_c\rangle$, $|1\rangle|E_c\rangle$,... , $|n_m\rangle|E_c\rangle$, and $|0\rangle|E_v\rangle$, $|1\rangle|E_v\rangle$,... , $|n_m\rangle|E_v\rangle$.

4.1 Two-Level System and Single-Mode Photons

The Hamiltonian is

$$\mathcal{H} = \hbar\omega a^\dagger a + \begin{pmatrix} E_c & 0 \\ 0 & E_v \end{pmatrix} - \mathbf{E}\cdot\mathbf{d}. \tag{74}$$

where the electric field operator is

$$\mathbf{E} = \frac{\boldsymbol{\mathcal{E}}_\omega a + \boldsymbol{\mathcal{E}}_\omega^* a^\dagger}{2}, \tag{75}$$

and the dipole matrix operator is

$$\begin{pmatrix} 0 & \mathbf{d}_{cv} \\ \mathbf{d}_{vc} & 0 \end{pmatrix}, \tag{76}$$

where we assume that the diagonal terms vanish. The transition rate from $|n\rangle|E_c\rangle$ to $|n+1\rangle|E_v\rangle$ is obtained by

$$W_{\text{emission}} = \frac{\pi}{2}\frac{|\langle n+1|\langle E_v|\mathbf{H}_I|n\rangle|E_c\rangle|^2}{\hbar^2}\delta(\omega-\omega_{cv}) \tag{77}$$

$$= \frac{(n+1)\pi}{2}\frac{|\boldsymbol{\mathcal{E}}_\omega\cdot\mathbf{d}_{cv}|^2}{\hbar^2}\delta(\omega-\omega_{cv}). \tag{78}$$

An interesting result occurs when $n = 0$. The emission is not zero when $n = 0$. This is the phenomenon "spontaneous emission". When $n > 0$, it corresponds to the stimulated emission. The transition rate from $|n\rangle|E_v\rangle$ to $|n-1\rangle|E_c\rangle$ is obtained by

$$W_{\text{absorption}} = \frac{\pi}{2}\frac{|\langle n-1|\langle E_c|\mathbf{H}_I|n\rangle|E_v\rangle|^2}{\hbar^2}\delta(\omega-\omega_{cv}) \tag{79}$$

$$= \frac{n\pi}{2}\frac{|\boldsymbol{\mathcal{E}}_\omega^*\cdot\mathbf{d}_{vc}|^2}{\hbar^2}\delta(\omega-\omega_{cv}). \tag{80}$$

4.2 Jaynes–Cummings Model

The TLS and single-mode photon Hamiltonian can be further simplified with the RWA,

The Hamiltonian is

$$\mathcal{H} = \hbar\omega a^\dagger a + \begin{pmatrix} E_c & 0 \\ 0 & E_v \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & \boldsymbol{\mathcal{E}}_\omega \cdot \mathbf{d}_{cv} a + \boldsymbol{\mathcal{E}}^*_\omega \cdot \mathbf{d}_{cv} a^\dagger \\ \boldsymbol{\mathcal{E}}_\omega \cdot \mathbf{d}_{vc} a + \boldsymbol{\mathcal{E}}^*_\omega \cdot \mathbf{d}_{vc} a^\dagger & 0 \end{pmatrix} \tag{81}$$

$$\simeq \hbar\omega a^\dagger a + \begin{pmatrix} E_c & 0 \\ 0 & E_v \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & \boldsymbol{\mathcal{E}}_\omega \cdot \mathbf{d}_{cv} a \\ \boldsymbol{\mathcal{E}}^*_\omega \cdot \mathbf{d}_{vc} a & 0 \end{pmatrix} \tag{82}$$

$$= \hbar\omega a^\dagger a + \frac{E_c + E_v}{2} + \frac{\hbar\omega_{cv}}{2}\sigma_z + \hbar\left(\lambda\sigma_+ a + \lambda^* \sigma_- a^\dagger\right) \tag{83}$$

where

$$\lambda = \frac{-\boldsymbol{\mathcal{E}}_\omega \cdot \mathbf{d}_{cv}}{2\hbar}. \tag{84}$$

The average energy $\frac{E_c+E_v}{2}$ is only a constant so as irrelavent to dynamics. In most cases, it is possible to make λ real by choosing the phase of $\mathbf{d}_{cv}$. The Jaynes–Cummings Model is then obtained as

$$\mathcal{H}_{JC} = \hbar\omega a^\dagger a + \frac{\hbar\omega_{cv}}{2}\sigma_z + \hbar\lambda\left(\sigma_+ a + \sigma_- a^\dagger\right). \tag{85}$$

We have used the Pauli matrices

$$\sigma_z = |E_c\rangle\langle E_c| - |E_v\rangle\langle E_v| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{86}$$

$$\sigma_+ = |E_c\rangle\langle E_v| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{87}$$

$$\sigma_- = |E_v\rangle\langle E_c| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{88}$$

The electron number operator is an identity,

$$N_e = |E_c\rangle\langle E_c| + |E_v\rangle\langle E_v|, \tag{89}$$

and the excitation number operator is

$$N_{ex} = |E_c\rangle\langle E_c| + a^\dagger a. \tag{90}$$

These numbers are conservative since the commutators vanish

$$[H, N_e] = 0, \tag{91}$$

$$[H, N_{ex}] = 0. \tag{92}$$

Exercise 1: Excitation Number

Show Eq. (92).

References

[1] J. J. Sakurai, *Modern Quantum Mechanics*, 1994 and 2010