Phase Space Descriptions in the Coherent State Basis

Jhih-Sheng Wu
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1.1 Phase Space Pictures

The state of a classical particle is fully determined by its x and p. A useful way to represent the states is the phase space (x, p), where the horizontal axis is x and the vertical axis is p. A state of a classical particle is one point in the phase phase. The time evolution of a state is the trajectory in the phase space. The trajectory $(x(t), p(t))$ contains all the information of the particle. The classic example is the harmonic oscillator with

$$x(t) = x_0 \cos(\omega t + \phi), \tag{1}$$
$$p(t) = -\omega x_0 \sin(\omega t + \phi), \tag{2}$$

or in the dimensionless expression

$$\tilde{x}(t) = \frac{x(t)}{x_0} = \cos(\omega t + \phi), \tag{3}$$
$$\tilde{p}(t) = \frac{p(t)}{\omega x_0} = -\sin(\omega t + \phi). \tag{4}$$

The state travels along the trajectory is a unit circle (see Fig. 1).

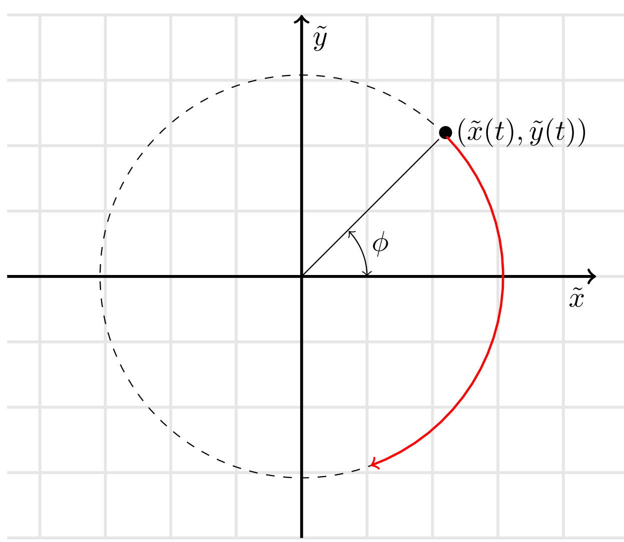

Figure 1: A classical state is a point in the phase space. The motion of a state is a trajectory. In the case of a harmonic oscillator, the trajectory is a circle.

An ensemble of classical particles are described by the phase space probability density function $f(x,p)$, where to find a particle with a position x and a momentum p is given by

$$f(x,p)dxdp, \tag{5}$$

and the normalization condition is

$$\int dx \int dp f(x,p)dxdp = 1. \tag{6}$$

Classically, the function $f(x,p)$ of a pure state, i.e., a single particle, is a delta function $f(x,p) = \delta(x-x_0)\delta(p-p_0)$. We have make the analogies $x \leftrightarrow X$ and $y \leftrightarrow Y$. One question arises: can we define a function similar to $f(x,p)$ to describe states or ensembles of photons? The problem is that a quantum state can not have exact X and Y at the same time. Thus, a quantum state is not a single point in the phase space. Recall the relations

$$X = \frac{a+a^\dagger}{2}, \tag{7}$$

$$Y = \frac{a-a^\dagger}{2i}. \tag{8}$$

For a coherent state $|\alpha\rangle$, we have the relations

$$\langle X\rangle = \frac{\alpha+\alpha^*}{2}, \tag{9}$$

$$\langle Y\rangle = \frac{\alpha-\alpha^*}{2i}. \tag{10}$$

As you can show $\sigma(X) = \sigma(Y) = 1/2$ for a coherent state, it means that a state in the phase space is not a point by a blurred circular cloud (see Fig. 2). The size of the cloud reflects the uncertainty relations. Coherent states are the states satisfying the minimum uncertainty relations. In general, an arbitrary state can have a very broad distribution in the phase space.

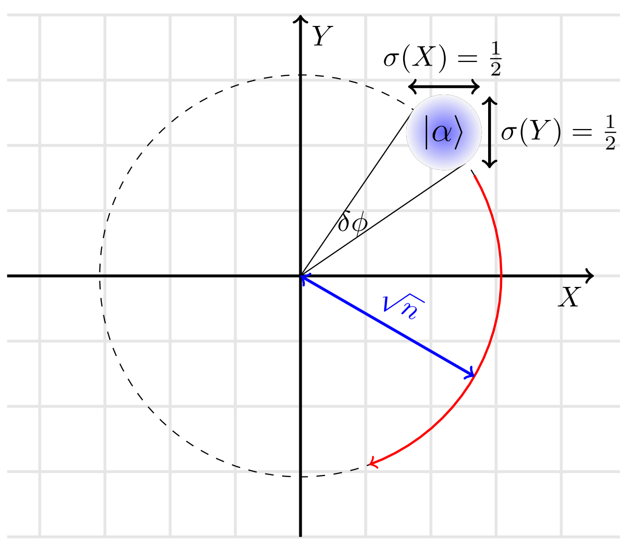

Figure 2: A coherent state is a fuzzy circle in the phase space.

A mapping of a state $|\psi\rangle$ or an ensemble to a distribution in the phase space (X, Y) (or equivalently the complex α space.) provide a physical picture. However, a mapping $|\psi\rangle \to f(X, Y)$ is not uniquely defined. The problem comes from that X and Y are non-commutative operators. There exist many attempts to define a probability density $f(X, Y)$ or $f(\alpha)$. We are going to introduce the three most used definitions,

- Wigner distribution
- Q-function
- P-function

Note that the definitions and calculations of these functions are quite mathematically involved. These functions serve as quantitative tools to describe the phase space probability densities. It is fine to have a qualitative picture in mind first and know more calculations when it is needed.

1.2 Properties of Coherent States

1.2.1 Orthogonality

Two coherent states $|\alpha\rangle$ and $|\beta\rangle$ are not orthogonal,

$$\begin{aligned}
\langle\beta|\alpha\rangle &= e^{-\frac{(|\alpha|^2+|\beta|^2)}{2}} \sum_{n,m} \frac{(\beta^*)^m(\alpha)^n}{\sqrt{m!n!}} \langle m|n\rangle && (11)\\
&= e^{-\frac{(|\alpha|^2+|\beta|^2)}{2}} \sum_{n} \frac{(\beta^*)^n(\alpha)^n}{n!} && (12)\\
&= e^{-\frac{(|\alpha|^2+|\beta|^2)}{2}} e^{\beta^*\alpha} && (13)\\
&= e^{-\frac{|\alpha-\beta|^2}{2}} e^{\frac{\beta^*\alpha-\beta\alpha^*}{2}}, && (14)
\end{aligned}$$

which does not vanish.

1.2.2 Identity

The identity can be expressed with the coherent states,

$$\int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha| \equiv \int \frac{d\mathrm{Re}[\alpha]d\mathrm{Im}[\alpha]}{\pi}|\alpha\rangle\langle\alpha| = \mathbb{1}. \tag{15}$$

Derivation 1: Identity with Coherent States

The proof of Eq. (15) is as follows. Let $\alpha = re^{\phi}$ and.

$$d\alpha^2 = d\mathrm{Re}[\alpha]d\mathrm{Im}[\alpha] = rdrd\theta. \tag{16}$$

The left hand side of Eq. (15) becomes

$$\int \frac{rdrd\theta}{\pi}|\alpha\rangle\langle\alpha| = \int \frac{rdrd\theta}{\pi}e^{-r^2}\sum_{m,n}\frac{e^{i(n-m)\theta}r^{m+n}}{n!}|m\rangle\langle n| \tag{17}$$

$$= \sum_n \frac{\int dre^{-r^2}2r^{2n+1}}{n!}|n\rangle\langle n| \tag{18}$$

$$= \sum_n \frac{\int due^{-u}u^n}{n!}|n\rangle\langle n| \tag{19}$$

$$= \sum_n |n\rangle\langle n| = \mathbb{1}. \tag{20}$$

1.2.3 Coherent State Representations of Operators

Any operator X can be expressed in the coherent state bases with the identity Eq. (15),

$$X = \int\frac{d^2\alpha}{\pi}\int\frac{d^2\beta}{\pi}|\alpha\rangle\langle\alpha|X|\beta\rangle\langle\beta|. \tag{21}$$

However, coherent states are not orthogonal, so the coherent states form an **overcomplete** set of bases.[1] It is possible to write X in the coherent state diagonal form.

An operator X is uniquely determined $\langle\alpha|X|\alpha\rangle$. The diagonal element $\langle\alpha|A|\alpha\rangle$ in the number state basis is

$$\langle\alpha|X|\alpha\rangle = \exp -|\alpha|^2\sum_{m,n}\frac{\langle n|X|m\rangle\alpha^m(\alpha^*)^n}{\sqrt{m!n!}}, \tag{22}$$

indicating that $\langle\alpha|X|\alpha\rangle$ contains all the information of the elements $\langle n|X|m\rangle$, which forms a complete set.

Coherent state diagonal representation. Suppose that X has a series expansion of a and $a^\dagger$ in the antinormal ordering,

$$X = \sum_{mn}\chi^A_{nm}a^n(a^\dagger)^m, \tag{23}$$

[1] See Se. 5.4. of Ref. [1] for a more rigorous discussion.

where χ^A_{nm} is a c-number. The subscript A denotes the antinormal ordering. Inserting the identity, we obtain

$$X = \sum_{mn} \chi^A_{nm} a^n \left(\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| \right) (a^\dagger)^m \tag{24}$$

$$= \int d^2\alpha \, \chi^A(\alpha) |\alpha\rangle\langle\alpha| \tag{25}$$

where

$$\chi^A(\alpha) = \frac{1}{\pi} \sum_{mn} \chi^A_{nm} \alpha^n (\alpha^*)^m, \tag{26}$$

is a c-number.

2 Phase Space Distributions

Given a density matrix ρ, there are three important distribution functions which are the quantum analogs of the classical probability density $f(x, p)$.

2.1 Wigner Distribution

The Wigner function $W(\alpha)$ is defined as

$$W(\alpha) = \int \frac{d^2\eta}{\pi^2} e^{\eta^*\alpha - \eta\alpha^*} \chi_W(\eta), \tag{27}$$

where the characteristic function $\chi_W(\eta)$ is

$$\chi_W(\eta) = \mathrm{Tr}\left[\rho e^{\eta a^\dagger - \eta^* a}\right]. \tag{28}$$

Exercise 1: Normalization

Show that

$$\int \frac{d^2\alpha}{\pi^2} e^{\eta^*\alpha - \eta\alpha^*} = \delta_2(\eta) \equiv \delta(\mathrm{Re}[\eta])\delta(\mathrm{Im}[\eta]), \tag{29}$$

and use the result and Eq. (27) to show

$$\int d^2\alpha \, W(\alpha) = 1. \tag{30}$$

Hint: a delta function can be expressed as

$$\delta(x) = \frac{1}{2\pi}\int e^{ikx}dk. \tag{31}$$

The ensemble average of an operator X in this representation is

$$\langle X\rangle = \int d^2\alpha\,\chi^W(\alpha)W(\alpha), \tag{32}$$

where

$$\chi^W(\alpha) = \sum_{n,m}\chi^W_{nm}\alpha^n(\alpha^*)^m \tag{33}$$

The coefficient χ^W_{nm} is the Weyl(symmetric)-ordering representation of an operator X,

$$X = \sum_{m,n}\chi^W_{nm}\left(\frac{(a^\dagger)^n a^m + a^m(a^\dagger)^n}{2}\right). \tag{34}$$

2.2 Glauber–Sudarshan P-function

The P-function is defined by

$$\rho = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|, \tag{35}$$

and satisfies the normalization condition

$$1 = \mathrm{Tr}[\rho] = \int d^2\alpha P(\alpha). \tag{36}$$

The P-function can be obtained by the normal-ordering characteristic function

$$P(\alpha) = \int \frac{d^2\eta}{\pi^2}e^{\eta^*\alpha-\eta\alpha^*}\chi_N(\eta), \tag{37}$$

where the characteristic function $\chi_N(\eta)$ is

$$\chi_N(\eta) = \mathrm{Tr}\left[\rho e^{\eta a^\dagger}e^{-\eta^* a}\right]. \tag{38}$$

The ensemble average of an operator X in this representation is

$$\langle X\rangle = \int d^2\alpha\,\chi^N(\alpha)P(\alpha), \tag{39}$$

where

$$\chi^N(\alpha) = \sum_{n,m} \chi^N_{nm} \alpha^n (\alpha^*)^m \tag{40}$$

The coefficient χ^N_{nm} is the normal-ordering representation of an operator X,

$$X = \sum_{m,n} \chi^N_{nm} (a^\dagger)^m a^n. \tag{41}$$

Note 1: Classical States and Nonclassical States

A state with $P(\alpha) < 0$ is defined as a nonclassical state.

2.3 Q-function

The Q-function is defined by

$$Q(\alpha) = \frac{1}{\pi} \langle \alpha | \rho | \alpha \rangle, \tag{42}$$

which is always positive since it is the diagonal element of the density matrix. The function $Q(\alpha)$ satisfies

$$0 \le Q(\alpha) \le \frac{1}{\pi}, \tag{43}$$

and

$$\mathrm{Tr}[\rho] = \int d^2\alpha Q(\alpha) = 1. \tag{44}$$

The Q-function can be obtained by the antinormal-ordering characteristic function

$$Q(\alpha) = \int \frac{d^2\eta}{\pi^2} e^{\eta^* \alpha - \eta \alpha^*} \chi_A(\eta), \tag{45}$$

where the characteristic function $\chi_A(\eta)$ is

$$\chi_A(\eta) = \mathrm{Tr}\left[\rho e^{-\eta^* a} e^{\eta a^\dagger}\right]. \tag{46}$$

The ensemble average of an operator X in this representation is

$$\langle X \rangle = \int d^2\alpha \chi^A(\alpha) Q(\alpha), \tag{47}$$

	$W(\alpha)$	$Q(\alpha)$	$P(\alpha)$
coherent state $\|\alpha_0\rangle$	$\frac{2}{\pi}e^{-2\|\alpha-\alpha_0\|^2}$	$\frac{1}{\pi}e^{-\|\alpha-\alpha_0\|^2}$	$\delta^2(\alpha-\alpha_0)$
thermal ensemble	$\frac{1}{\pi(\bar{n}+1/2)}\exp\left(-\frac{\|\alpha\|^2}{\bar{n}+1/2}\right)$	$\frac{1}{\pi(\bar{n}+1)}\exp\left(-\frac{\|\alpha\|^2}{\bar{n}+1}\right)$	$\frac{1}{\pi(\bar{n})}\exp\left(-\frac{\|\alpha\|^2}{\bar{n}}\right)$
pure number ensemble $\|1\rangle\langle 1\|$	$-(1-4\|\alpha\|^2)\frac{2}{\pi}e^{-2\|\alpha\|^2}$	$\frac{\|\alpha\|^2}{\pi}e^{-\|\alpha\|^2}$	singular

Table 1: Examples of $W(\alpha)$, $Q(\alpha)$, and $P(\alpha)$

where

$$\chi^A(\alpha) = \sum_{n,m} \chi^A_{nm} \alpha^n (\alpha^*)^m. \tag{48}$$

The coefficient χ^A_{nm} is the antinormal-ordering representation of an operator X,

$$X = \sum_{m,n} \chi^A_{nm} a^n (a^\dagger)^m. \tag{49}$$

Summary 1: Coherent States

(a) The phase space of photon states or ensembles are described by the two dimensional complex α plane.

(b) The real part and imaginary part of α are related to the quadrature operator X and Y.

$$X = \mathrm{Re}[\alpha], \tag{50}$$
$$Y = \mathrm{Im}[\alpha]. \tag{51}$$

(c) A coherent state $|\alpha_0\rangle$ is a fuzzy circle on the complex α plane.

(d) The coherent states are not orthogonal, so they are overcomplete.

(e) There are three ways to write the probability density

- Wigner distribution $W(\alpha)$: symmetric ordering
- Q-function $Q(\alpha)$: antinormal ordering
- P-function $P(\alpha)$: normal ordering

References

[1] J. C. Garrison and R. Y. Chiao, *Quantum Optics*, Oxford University Press 2008