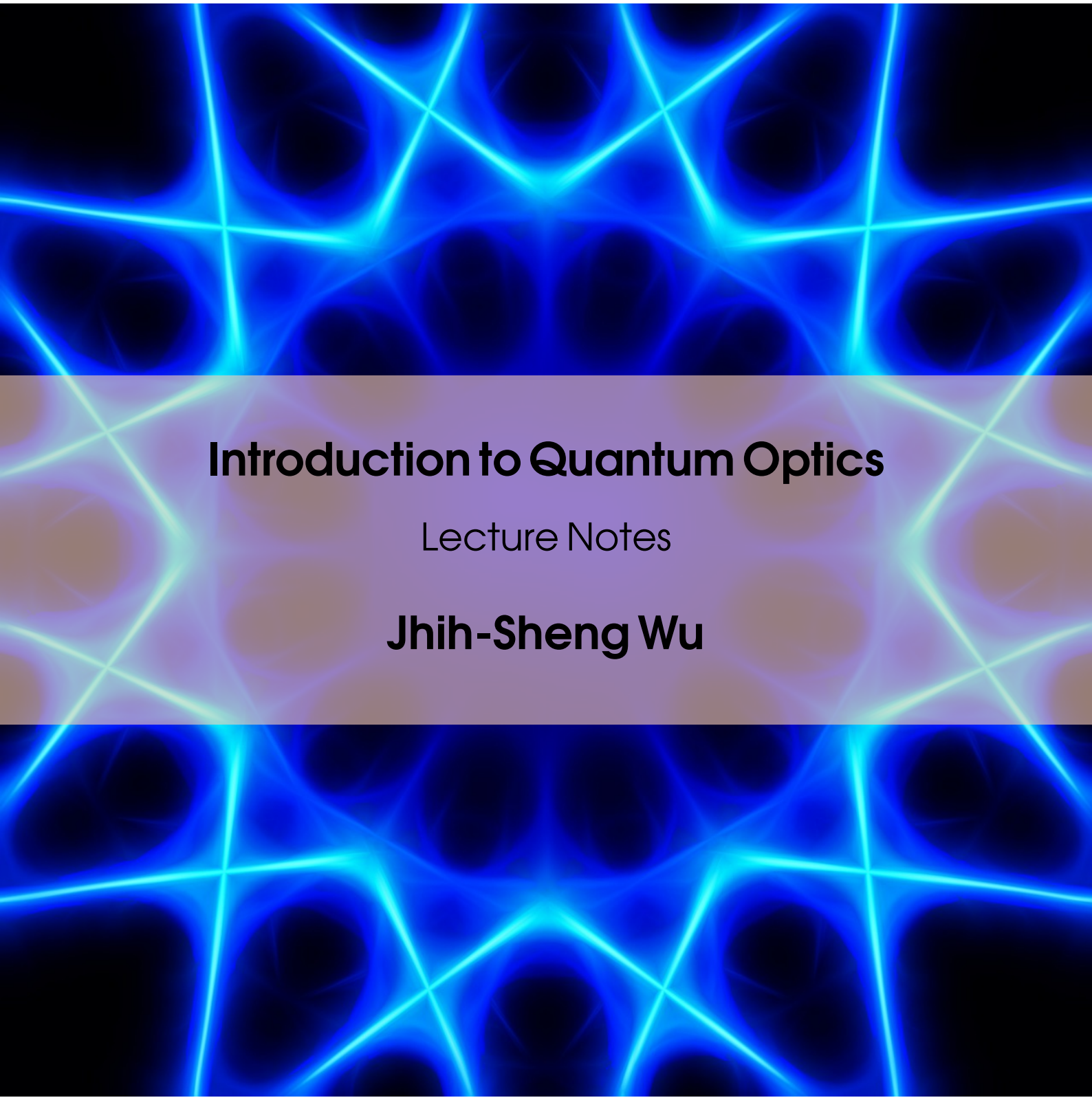

Introduction to Quantum Optics

Lecture Notes

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1. Review of Quantum Mechanics

Quantum mechanics is a modern mathematical theory used to describe the quantum phenomena. Although many scientists think it is not the ultimate theory, it is the most accurate theory today that describes the experiments. Quantum mechanics is formulated under the postulates, which are derived after many attempts to explain the experiments. In classical mechanics, a physical system consists of physical quantities which have definite values. For examples, the position x and the momentum p of a particle at any given time t are assumed to be some numbers. On the contrary, a physical system in quantum mechanics is described by a state $|\psi\rangle$. The notation $|\psi\rangle$ is called a ket. In a closed system, the state $|\psi\rangle$ contains all the information of the systems. The exotic part of quantum mechanics is that even $|\psi\rangle$ is complete, the outcome of observed quantities are still probabilistic.

1.1 Wavefunction

Let's use the wavefunction to elaborate the nature of probability. The wavefunction of a particle is obtained by writing $|\psi\rangle$ in the x basis $|x\rangle$,

$$\psi(x) \equiv \langle x|\psi\rangle. \tag{1.1.1}$$

For a given wavefunction $\psi(x)$, the probability to find the particle to be at x is $|\psi(x)|^2 dx$. Since the total probability is one, the normalization of a state requires that

$$\int |\psi(x)|^2 dx = 1. \tag{1.1.2}$$

The average position $\langle x\rangle$ (expectation value) of the particle is

$$\langle x\rangle = \int x|\psi(x)|^2 dx. \tag{1.1.3}$$

With the definitions, the well known Schrödinger's equation reads

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + V(x)\psi. \tag{1.1.4}$$

The representation of a state $|\psi\rangle$ is not unique. For example, we can use the momentum basis $|p\rangle$ to write $|\psi\rangle$,

$$\phi(p) \equiv \langle p|\psi\rangle, \tag{1.1.5}$$

and

$$\int |\phi(p)|^2 dp = 1. \tag{1.1.6}$$

Eq. (1.1.4) is only one example of the Schrödinger's equation. We will learn more general approaches to write the equations of quantum mechanics. There, we will start from the **Hamiltonian** of a system.

1.2 Dirac Notations

In quantum mechanics, the Bra-Ket notations are convenient tools. Any states are written as Kets $|\psi_1\rangle$, $|\psi_2\rangle$, $|\psi_3\rangle$,... You can think a Ket as a column vector. However, the representation of a column vector depends on the bases. For example, in the position basis, Kets can be defined as:

$$|\psi\rangle = \begin{pmatrix} \psi(x_1) \\ \psi(x_2) \\ \vdots \\ \psi(x_N) \end{pmatrix}. \tag{1.2.1}$$

whereas in the momentum basis

$$|\psi\rangle = \begin{pmatrix} \phi(p_1) \\ \phi(p_2) \\ \vdots \\ \phi(p_N) \end{pmatrix} \tag{1.2.2}$$

The role of a Bra is similar to row vectors in linear algebra. In the position basis, Bra can be defined as:

$$\langle\psi| = \begin{pmatrix} \psi^*(x_1) & \psi^*(x_2) & \dots & \psi^*(x_N) \end{pmatrix}. \tag{1.2.3}$$

whereas in the momentum basis

$$\langle\psi| = \begin{pmatrix} \phi^*(p_1) & \phi^*(p_2) & \dots & \phi^*(p_N) \end{pmatrix}. \tag{1.2.4}$$

The inner product of two states $|\psi\rangle$ and $|\phi\rangle$ is

$$\langle\psi|\phi\rangle, \tag{1.2.5}$$

which is a complex number. The inner product $\langle\psi_i|\psi_i\rangle$ is the probability to find the particle in the ith state. The outer product of two states $|\psi\rangle$ and $|\phi\rangle$ is

$$|\phi\rangle\langle\psi| \tag{1.2.6}$$

which is a matrix.

Exercise 1.1 Calculation of bras and ketscbk Let

$$|a\rangle = \begin{pmatrix} 1 \\ 2i \\ 3 \end{pmatrix} \tag{1.2.7}$$

$$|b\rangle = \begin{pmatrix} i \\ 0 \\ 2 \end{pmatrix} \tag{1.2.8}$$

What are $\langle a|$ and $\langle b|$?

2. Calculate $\langle a|a\rangle$, $\langle b|b\rangle$, $\langle a|b\rangle$ and $\langle b|a\rangle$?
3. Calculate $|a\rangle\langle b|$ and $|b\rangle\langle a|$. Are they complex conjugate of each other?

■

If $|\psi\rangle$ is to describe a single particle, the normalization of a state requires the inner product

$$\langle\psi|\psi\rangle = 1 \tag{1.2.9}$$

or in a specific basis

$$\sum_i |\psi_i|^2 = 1, \tag{1.2.10}$$

and for a continuous variable like x,

$$\int dx|\psi(x)|^2 = 1. \tag{1.2.11}$$

In the position basis, the position is a operator $\hat{x}$ (a matrix).

$$\hat{x} = \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_N \end{pmatrix} \tag{1.2.12}$$

As it should be, the operator $\hat{x}$ is a diagonal matrix in the position basis. In the Dirac's notation, the expectation value of x is

$$\langle x\rangle = \langle\psi|\hat{x}|\psi\rangle \tag{1.2.13}$$

$$= \sum_i x_i|\psi_i|^2 \tag{1.2.14}$$

$$= \int x|\psi(x)|^2 dx. \tag{1.2.15}$$

1.3 Postulates of Quantum Mechanics

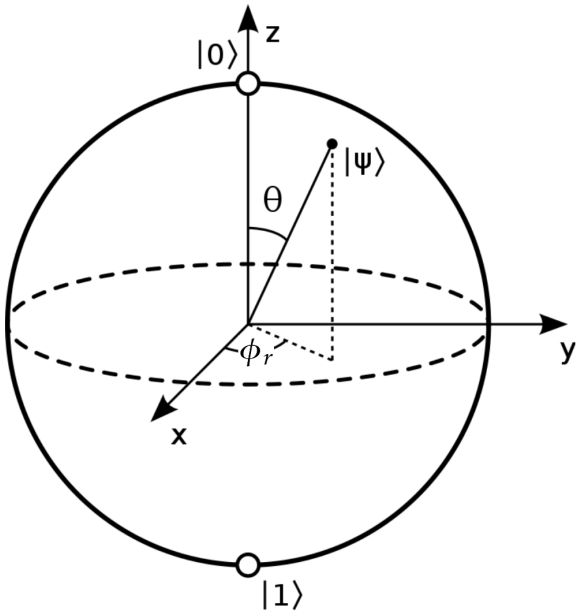

Figure 1.1: Bloch sphere.

Postulate 1.1 — State Vector. A physical system is completely described by a **complex** state vector $|\psi\rangle$ in the Hilbert space.

The state vector $|\psi\rangle$ contains all the information. The state vector can be written as a sum of other (basis) vectors.

$$|\psi\rangle = \sum_i \alpha_i |\psi_i\rangle \tag{1.3.1}$$

The probability to find the system in the ith state is $|\alpha_i|^2$. The simplest example is the qubit,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle. \tag{1.3.2}$$

Without losing the generality, the qubit can be written as ($|\alpha|^2 + |\beta|^2 = 1$)

$$|\psi\rangle = e^{i\phi_g}\left(\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi_r}|1\rangle\right), \tag{1.3.3}$$

where ϕ_g is the global phase, and ϕ_r is the relative phase between the $|0\rangle$ and $|1\rangle$ states. Without comparing with another qubit, the phase ψ_g does not have much meaning. The degrees of freedoms of a qubit are given by θ and ϕ_r, which correspond to a surface of a sphere. The space of a qubit is called the Bloch sphere.

Postulate 1.2 — Temporal Evolution. The evolution of a closed quantum state is described by the unitary transformation.

The state $|\psi(t')$ is related to the state $|\psi(t)$ by

$$|\psi(t')\rangle = \hat{U}(t,t')|\psi(t)\rangle, \tag{1.3.4}$$

where $U(t,t')$ is a unitary operator (a matrix), $U^\dagger U = \mathbb{1}$. [1]

The postulate comes from the conservation of total probabilities,

$$\langle\psi(t')|\psi(t')\rangle = \langle\psi(t)|\psi(t)\rangle = 1 \tag{1.3.5}$$

The unitary operator can be written as

$$U(t,t') = e^{-i\frac{\mathcal{H}}{\hbar}(t'-t)}, \tag{1.3.6}$$

where $\mathcal{H}$ has to be hermitian, $\mathcal{H} = \mathcal{H}^\dagger$, to make $U(t',t)$ unitary.

Exercise 1.2 Exponential Function of Matricesefm Show that the operator defined by Eq. (1.3.6) is unitary. Use the following facts,

The matrix exponential of a matrix M is defined $e^M = \mathbb{1} + M + \frac{M^2}{2!} + \ldots$

- $e^A e^B = e^{A+B}$ if $[A,B] \equiv AB - BA = 0$. This can be proved by using the above definition. This equation is a special case of the Baker–Campbell–Hausdorff formula, which reads

$$e^X e^Y = e^Z$$
$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots,$$

- $\mathcal{H} = \mathcal{H}^\dagger$

■

The postulate gives rise to the general form of the time-dependent Schrödinger's equation,

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \mathcal{H}|\psi\rangle. \tag{1.3.7}$$

The operator $\mathcal{H}$ is called the "**Hamiltonian**" of the system. The Hamiltonian, coming from the classical mechanics, typically is the total energy of the system. For example, for a particle, the Hamiltonian $\mathcal{H} = \frac{p^2}{2m} + V(x)$. The case of a particle is only one of the examples. If the systems are discrete and finite (energy levels), the Hamiltonian is a finite-dimensional matrix. For example, classically, the energy of a magnetic moment $\boldsymbol{\mu}$ in a magnetic filed $\mathbf{B}$ is $E = -\boldsymbol{\mu}\cdot\mathbf{B}$, and quantumly the Hamiltonian is $\mathcal{H} = -\boldsymbol{\mu}\cdot\mathbf{B}$, where the magnetic moment is related to the angular momentum operator $\mathbf{L}$ by $\boldsymbol{\mu} = \gamma\mathbf{L}$. In the case of an electron, the angular momentum operator is $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, where σ_i are the Pauli matrices (2 by 2 matrices). Hence, the dimension of the Hamiltonian is two.

Let's consider a system with N levels of the energies E_1, E_2, ..., E_N. The energy eigenstates, $|E_i\rangle$, satisfy

$$\mathcal{H}|E_i\rangle = E_i|E_i\rangle. \tag{1.3.8}$$

The Hamiltonian in the energy bases $|E_i\rangle$ is diagonal

$$\mathcal{H} = \begin{pmatrix} E_1 & 0 & 0 & 0 \\ 0 & E_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_N \end{pmatrix}. \tag{1.3.9}$$

[1] For the sake of simplicity, I won't use a hat for an operator all the time unless there will be confusion.

The solution of the time-dependent Schrödinger's equation (Eq. (1.3.7)) is

$$|\psi(t)\rangle = \sum_i \alpha_i e^{-i\frac{E_i}{\hbar}t}|E_i\rangle, \tag{1.3.10}$$

where α_i are the coefficients of the initial state in terms of $|E_i\rangle$.

Postulate 1.3 — Measurement. Quantum measurement (collapse). A measure makes a system $|\psi\rangle$ collapse into some state $|\psi_i\rangle$. The possible outcome states $|\psi_i\rangle$ depend on the measurements. For example, if we measure the position of a particle, the outcome states are $|x\rangle$ with $-\infty < x < \infty$. A measurement is described by s set of operators $\{M_m\}$, where m denotes all the possible outcome states. After a measure measurement, the state become

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}} \tag{1.3.11}$$

with the probability

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle. \tag{1.3.12}$$

The completeness theorem requires that

$$\sum_m M_m^\dagger M_m = \mathbb{1}. \tag{1.3.13}$$

For example, the measurement operator on a qubit are

$$M_0 = |0\rangle\langle 0| \tag{1.3.14}$$
$$M_1 = |1\rangle\langle 1| \tag{1.3.15}$$

Exercise 1.3 — Qubit Measurement. The initial qubit state is $\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$.

1. What are the two possible states after a measurement of $\{M_0, M_1\}$?
2. What are the probabilities to be the above two states, respectively?

■

Heisenberg uncertainty principle

Postulate 1.4 — Hermiticity. Any physical observables are Hermitian operators. For example, in the position basis, the position and momentum operators are

$$\hat{x} = x \tag{1.3.16}$$
$$\hat{p} = -i\hbar\frac{\partial}{\partial x} \tag{1.3.17}$$

Let A be the physical observable operator. The expectation value of A of a state $\psi\rangle$ is

$$\langle A\rangle = \langle\psi|A|\psi\rangle. \tag{1.3.18}$$

The state $|p\rangle$ is the eigenvector of the momentum operator $\hat{p}$,

$$\hat{p}|p\rangle = p|p\rangle, \tag{1.3.19}$$

and for the position operator $\hat{x}$,

$$\hat{x}|x\rangle = x|x\rangle. \tag{1.3.20}$$

Note that the eigenvectors of a Hermitian operator form a complete set of bases of the space.

The eigenvectors $|A_i\rangle$ of A forms a complete set of bases of the state space. The eigenstates are orthogonal and normal,

$$\langle A_j|A_i\rangle = \delta_{ij}. \tag{1.3.21}$$

That is, any state $|\psi\rangle$ can be written as

$$|\psi\rangle = \sum_i \alpha_i |A_i\rangle. \tag{1.3.22}$$

The completeness implies that the identity $\mathbb{1}$ is,

$$\mathbb{1} = \sum_i |A_i\rangle\langle A_i| \tag{1.3.23}$$

The standard deviation of A is $\sigma(A)$,

$$\sigma(A) \equiv \sqrt{\langle A^2\rangle - \langle A\rangle^2}. \tag{1.3.24}$$

Two operators A and B are compatible if their commutator $[A,B] \equiv AB - BA = 0$. Otherwise, they are incompatible. If

$$[A,B] = c \tag{1.3.25}$$

and c is a number, the general uncertainty principle reads

$$\sigma(A)\sigma(B) \geq \frac{|\langle\psi|[A,B]|\psi\rangle|}{2} = \frac{|c|}{2}. \tag{1.3.26}$$

Exercise 1.4 Uncertainty Principleuncer Prove the Heisenberg uncertainty principle, Eq. (1.3.26). Hint: use the Cauchy–Schwarz inequality.

$$\langle\psi|\psi\rangle\langle\phi|\phi\rangle \geq |\langle\psi|\phi\rangle|^2, \tag{1.3.27}$$

where $|\psi\rangle$ and $|\phi\rangle$ are two states. ∎

The most classical example of the uncertainty principle is about x and p,

$$[x,p] = i\hbar. \tag{1.3.28}$$

The uncertainty principle reads

$$\sigma(x)\sigma(p) \geq \frac{\hbar}{2}. \tag{1.3.29}$$

1.4 Quantum Dynamics: Schrödinger, Interaction, Heisenberg Pictures

In the experiments, we are interested in the dynamics of ant observable A, more specifically, the expectation

$$\langle A(t)\rangle = \langle\psi(t)|A|\psi(t)\rangle \tag{1.4.1}$$

There are three main pictures to interpret and solve the problem.

1.4.1 Schrödinger Picture

Consider that the observable operator A is static and the states $|\psi(t)\rangle$ is evolving.

$$|\psi(t)\rangle \equiv |\psi(t)\rangle_S = e^{-\frac{i\mathcal{H}t}{\hbar}}|\psi(0)\rangle \tag{1.4.2}$$

The expectation value in this picture is

$$\langle A(t)\rangle = \langle\psi(t)|A|\psi(t)\rangle \tag{1.4.3}$$

1.4.2 Heisenberg Picture

Consider that the observable operator $A(t)$ is dynamic and the states $|\psi(t)\rangle$ is static.

$$A_h \equiv A(t) = e^{\frac{i\mathcal{H}t}{\hbar}} A e^{-\frac{i\mathcal{H}t}{\hbar}}, \tag{1.4.4}$$

and the expectation value is

$$\langle A(t)\rangle = \langle\psi(0)|A_h|\psi(0)\rangle. \tag{1.4.5}$$

The evolution of A_h follows the Heisenberg's equation,

$$i\hbar\frac{\partial A_h}{\partial t} = [A_h, \mathcal{H}]. \tag{1.4.6}$$

Exercise 1.5 Proof of the Heisenberg's equationphe Let $U(t) = e^{-\frac{i\mathcal{H}t}{\hbar}}$ so that $A_h = U^\dagger A U$. Differentiating A_h with respect to t gives

$$\frac{\partial A_h}{\partial t} = \frac{\partial U^\dagger}{\partial t} A U + U^\dagger A \frac{\partial U}{\partial t} \tag{1.4.7}$$

First, show that the derivative of $U(t)$ is

$$i\hbar\frac{\partial}{\partial t}U(t) = \mathcal{H}U(t). \tag{1.4.8}$$

Use the two above equations to prove the Heisenberg's equation. ■

1.4.3 Interaction Picture

When the Hamiltonian includes two terms: one is the original Hamiltonian $\mathcal{H}_0$ and the interaction with the external system $V(t)$, it is convenient to use the interaction picture, where both the states and the operator are evolving. The total Hamiltonian is $\mathcal{H} = \mathcal{H}_0 + V(t)$. The state $|\psi\rangle_I$ is

$$|\psi\rangle_I = e^{i\frac{\mathcal{H}_0 t}{\hbar}}|\psi(t)\rangle_S, \tag{1.4.9}$$

and the operator A_I is

$$A_I = e^{i\frac{\mathcal{H}_0 t}{\hbar}} A e^{-i\frac{\mathcal{H}_0 t}{\hbar}}, \tag{1.4.10}$$

The Schrödinger equation becomes

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle_I = V_I(t)|\psi\rangle_I, \tag{1.4.11}$$

$$V_I(t) \equiv e^{i\frac{\mathcal{H}_0 t}{\hbar}} V(t) e^{-i\frac{\mathcal{H}_0 t}{\hbar}}. \tag{1.4.12}$$

Note that the solution to Eq. (1.4.11) is not $|\psi(t)\rangle_I = e^{-i\frac{\mathcal{H}_I}{\hbar}t}|\psi(0)\rangle_I$ because the $V_I(t)$ is time-dependent. The solution to to Eq. (1.4.11) is

$$|\psi(t)\rangle_I = U_I(t,t_0)|\psi(t_0)\rangle_I \tag{1.4.13}$$

$$U_I(t,t_0) = \mathbb{1} - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t')U_I(t',t_0)dt' \tag{1.4.14}$$

The Heisenberg's equation becomes

$$i\hbar\frac{\partial A_I}{\partial t} = [A_I, \mathcal{H}_0]. \tag{1.4.15}$$

1.5 Harmonic Oscillators

The Hamiltonian of a simple harmonic oscillator is

$$\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}, \tag{1.5.1}$$

where $\omega = \sqrt{k/m}$ and k is the spring constant. We define the creation operator $a^\dagger$ and the annihilation operator a,

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right), \tag{1.5.2}$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right). \tag{1.5.3}$$

Exercise 1.6 Commutation Relationcr Show that

$$[a, a^\dagger] = 1. \tag{1.5.4}$$

Use the relation $[x, p] = i\hbar$. ■

The Hamiltonian is rewritten as

$$\mathcal{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \tag{1.5.5}$$

$$= \hbar\omega(N + \frac{1}{2}) \tag{1.5.6}$$

where $N = a^\dagger a$ is the number operator. The eigenvector of N is $|n\rangle$

$$N|n\rangle = n|n\rangle, \tag{1.5.7}$$

where n is the eigenvalue. The number states are orthonormal

$$\langle m|n\rangle = \delta_{mn}. \tag{1.5.8}$$

Important indentities are

$$[N, a] = -a, \tag{1.5.9}$$

$$[N, a^\dagger] = a^\dagger, \tag{1.5.10}$$

As a result, we have

$$Na^\dagger|n\rangle = \left(a^\dagger N + a^\dagger\right)|n\rangle = (n+1)a^\dagger|n\rangle, \tag{1.5.11}$$

$$Na|n\rangle = (aN - a)|n\rangle = (n-1)a|n\rangle, \tag{1.5.12}$$

These eqautions imply that

$$a|n\rangle = c_-|n-1\rangle, \tag{1.5.13}$$

$$a^\dagger|n\rangle = c_+|n+1\rangle, \tag{1.5.14}$$

The constants c_- and c_+ can be fixed by noting that

$$\langle n|a^* a|n\rangle = n = |c_-|^2, \tag{1.5.15}$$

$$\langle n|aa^*|n\rangle = n + 1 = |c_+|^2. \tag{1.5.16}$$

Taking c_- and c_+ to be positive by convention, $c_- = \sqrt{n}$ and $c_+ = \sqrt{n+1}$. We have the important relations which explain the names, creation and annihilation,

$$a|n\rangle = \sqrt{n}|n-1\rangle, \tag{1.5.17}$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{1.5.18}$$

Note 1.1 — Representation in the number basis. The number n is the number of the energy quanta. The smallest number of n is $n = 0$. The physical meaning of $|n\rangle$ is a state containing n energy quanta. Thus, $|n\rangle$ is called the **number state**. The energy of a harmonic oscillator is

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \tag{1.5.19}$$

The $\frac{1}{2}\hbar\omega$ is interpreted as the vacuum energy since it exists even when $n = 0$. Applying a creation operator on the $|n\rangle$, the state $|n\rangle$ becomes $\sqrt{n+1}|n+1\rangle$, that is, the $a^\dagger$ will create one single quantum to the original state. Similarly, the a will annihilate one energy quantum from the system. We can also prove that

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle. \tag{1.5.20}$$

The position operator x and momentum operator p can be expressed as

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \tag{1.5.21}$$

$$p = i\sqrt{\frac{m\omega\hbar}{2}}(-a + a^\dagger) \tag{1.5.22}$$

1.5.1 Number States in the Position Bases

As the familiar wave function $\psi(x)$, we can express the $|n\rangle$ in the x bases. The wavefunctions are $\psi_n(x) \equiv \langle x|n\rangle$. Let's solve the ground states first $\psi_0(x)$. We start with

$$a|0\rangle = 0 \tag{1.5.23}$$

$$\Rightarrow \langle x|a|0\rangle = 0 \tag{1.5.24}$$

$$\Rightarrow \sqrt{\frac{m\omega}{2\hbar}}\left\langle x\left|x + \frac{ip}{m\omega}\right|0\right\rangle = 0 \tag{1.5.25}$$

$$\Rightarrow \left(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\psi_0(x) = 0 \tag{1.5.26}$$

$$\Rightarrow \psi_0(x) = \frac{1}{\pi^{1/4}\sqrt{x_0}}e^{-\frac{1}{2}\left(\frac{x}{x_0}\right)^2}, \tag{1.5.27}$$

where $x_0 = \sqrt{\frac{\hbar}{m\omega}}$

Exercise 1.7 Uncertainty of the ground stategs Show that for the ground state $\psi_0(x)$, the uncertainty relation has a equal sign, that is, the state has the minimum uncertainty,

$$\sigma(x)\sigma(p) = \frac{\hbar}{2}. \tag{1.5.28}$$

■

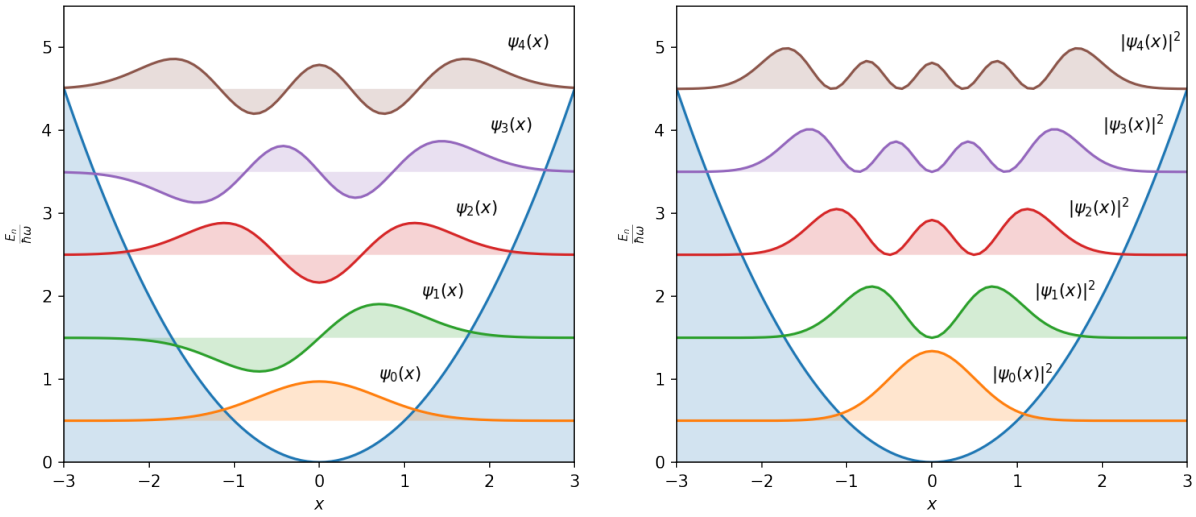

Figure 1.2: Wavefunction $\psi_n(x)$.

Using Eqs. (1.5.20) and (1.5.27), we obtain the expression for $\phi_n(x)$,

$$\psi_n(x) = \left(\frac{1}{\pi^{1/4}\sqrt{2^n n! x_0^{n+1/2}}}\right)\left(x - x_0^2\frac{\partial}{\partial x}\right)^n e^{-\frac{1}{2}\left(\frac{x}{x_0}\right)^2}. \tag{1.5.29}$$

1.5.2 Dynamics of a Harmonic Oscillator

The Heisenberg's Equations of a and $a^\dagger(t)$ are

$$i\hbar\frac{da^\dagger}{dt} = [a^\dagger, H] = -\hbar\omega a^\dagger, \tag{1.5.30}$$

$$i\hbar\frac{da}{dt} = [a, H] = \hbar\omega a, \tag{1.5.31}$$

$$\tag{1.5.32}$$

whose solutions are

$$a(t) = a(0)e^{-i\omega t}, \tag{1.5.33}$$

$$a^\dagger(t) = a^\dagger(0)e^{i\omega t}. \tag{1.5.34}$$

In terms of x and p, Eqs. (1.5.33) and (1.5.34) read

$$a(t) = x(t) + \frac{ip(t)}{m\omega} = \left(x(0) + \frac{ip(0)}{m\omega}\right)e^{-i\omega t}, \tag{1.5.35}$$

$$a^\dagger(t) = x(t) - \frac{ip(t)}{m\omega} = \left(x(0) - \frac{ip(0)}{m\omega}\right)e^{i\omega t}. \tag{1.5.36}$$

Solving the equations for $x(t)$ and $p(t)$, we have

$$x(t) = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t, \tag{1.5.37}$$

$$p(t) = -m\omega x(0)\sin\omega t + p(0)\cos\omega t. \tag{1.5.38}$$

Note 1.2 — Heisenberg picture of x and p of a harmonic oscillator. Equations (1.5.37) and (1.5.38) are exactly the same as the equations of motion derived from the classical mechanics. In contrast, $x(0)$ and $p(0)$ are operators. If we take the number state $|n\rangle$, the expectation value $\langle n|x(t)|n\rangle$ vanishes. We will not observe an expectation value $\langle x(t)\rangle$ obeying the classical motion. It turns out that the state mostly close to a classical state is the coherent state $|\lambda\rangle$, which is the eigenvector of the annihilation operator a,

$$a|\lambda\rangle = \lambda|\lambda\rangle. \tag{1.5.39}$$

We will talk more about the coherent states later.

1.6 Coherence and Decoherence at a Glance

Coherence refers to many meanings in different circumstances. We consider its usages in the context of physics. Roughly speaking, coherence means that two (or more than two) states (waves, particles) have a well defined correlation as time t or positions x change. For example, in the two-slit experiment, coherent light source is required to produce interference. Let the two waves through the two slits be $\psi_1(x_1,t_1)$ and $\psi_2(x_2,t_2)$. The fully coherence means that if we know the wave $\psi_1(x_1,t_1)$, we can know the $\psi_2(x_2,t_2)$ at any time and position. In this definition, it seems that every two sinusoidal waves $\exp(ik_1x-\omega_1t+\phi_1)$ and $\exp(ik_2x-\omega_2t+\phi_2)$ are coherent. However, we also require $k_1=k_2$ and $\omega_1=\omega_2$. The reason is as the following. When we measure the interference, frequently we collect the data for a long time over many periods . The interference signal is the an time average of the product $\psi_1^*(t)\psi_2(t)$. $|\psi_1(t)|^2$ and $|\psi_2(t)|^2$ are the background intensity. The time-averaged interference is

$$I_{\text{interference}} = \lim_{T\to\infty}\frac{\int_0^T 2\text{Re}\left[\psi_1^*(t)\psi_2(t)\right]}{T}. \tag{1.6.1}$$

If the two waves have different frequencies, the time-average vanishes.

Another question is that are any two waves fully coherent if they have the same frequencies. The answer is not necessary. Why? It is because the phase ψ_1 and ψ_2 can fluctuate. The coherence implies that $\delta=\phi_1-\phi_2$ is a constant as time t and position x changes. In practical situations, as the waves propagate, the environment provides noises to the phases. As a result, the time-average becomes smaller. This process is called "**decoherence**" Typically, a system gradually loses its coherence as t increases or traveled length x increases.

A more realistic system consists of many waves (states, particles)[2],

$$\psi(x,t)=\psi_1(x,t)+\psi_2(x,t)+\psi_3(x,t)+... \tag{1.6.2}$$

The interference involves all the cross-product terms $\psi_1^*(t)\psi_2(t)$, $\psi_2^*(t)\psi_3(t)$, $\psi_3^*(t)\psi_4(t)$, and so on. To deal with a system containing a large number of particles, it is more convenient to use a statistical tool than listing all the states. The idea is to use probabilities to describe distributions of states. This is the motivation of the density operator (matrix) formulation.

[2]We use the terms "particle", "waves", and "states" interchangeably.

1.7 Density Operator Formulation

Let's introduce the concepts of an ensemble.

Note 1.3 — Ensemble. An ensemble is a statistical tool to describe a system of many particles. An ensemble consists of a large number of virtual copies of a particle. Ideally, the number of states is infinite. Each copy represents a possible state that a particle can be in. A specific ensemble is specified by assigning the probability in each state.

For example, a photon state $|\text{photon}\rangle$ is decomposed as

$$|\text{photon}\rangle = \alpha|L\rangle + \beta|R\rangle, \tag{1.7.1}$$

where $|L\rangle$ ($|R\rangle$) is the left(right)-polarized state. In an ensemble, there are many photons. Let p_L and p_R be the probabilities of the left-polarized state and the right-polarized state, respectively, where

$$p_L + p_R = 1. \tag{1.7.2}$$

The probabilities p_L and p_R define the ensemble. We can not use the following expression to describe an ensemble,

$$\cancel{|\text{ensemble}\rangle = p_L|L\rangle + p_R|R\rangle,} \tag{1.7.3}$$

since this expression is used for one single state. The mathematical tool to describe an ensemble is the density matrix $\hat{\rho}$,

$$\hat{\rho} = p_L|L\rangle\langle L| + p_R|R\rangle\langle R|. \tag{1.7.4}$$

an ensemble is a statistical tool to describe a system of many particles. An ensemble consists of a large number of virtual copies of a particle. Ideally, the number of states is infinite. Each copy represents a possible state that a particle can be in. A specific ensemble is specified by assigning the probability in each state.

For example, a photon state $|\text{photon}\rangle$ is decomposed as

$$|\text{photon}\rangle = \alpha|L\rangle + \beta|R\rangle, \tag{1.7.5}$$

where $|L\rangle$ ($|R\rangle$) is the left(right)-polarized state. In an ensemble, there are many photons. Let p_L and p_R be the probabilities of the left-polarized state and the right-polarized state, respectively, where

$$p_L + p_R = 1. \tag{1.7.6}$$

1.7.1 Density Operators

In a general case, the density matrix $\hat{\rho}$ can be defined as

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \tag{1.7.7}$$

$$\sum_i p_i = 1, \tag{1.7.8}$$

$$0 \leq p_i \leq 1, \tag{1.7.9}$$

where $|\psi_i\rangle$ are the basis states, and p_i is the probability to find the particle at the ith state. Most times, $|\psi_i\rangle$ are chosen to be orthonormal vectors. In the definition by Eq. (1.7.7), the density matrix is a diagonal matrix. However, a density matrix can have nonzero off-diagonal elements. Consider a new set of orthonormal bases $|a_i\rangle$, obtained by the unitary transformation

$$|a_i\rangle = \sum_j U_{ij} |\psi_j\rangle. \tag{1.7.10}$$

where $U^\dagger U = \mathbb{1}$. The matrix element U_{ij} can be obtained explicitly by multiplying $\langle\psi_{j'}|$ on the both sides of Eq. (1.7.10),

$$U_{ij} = \langle\psi_j|a_i\rangle. \tag{1.7.11}$$

The inverse transforms are

$$|\psi_i\rangle = \sum_j U_{ij}^\dagger |a_j\rangle, \tag{1.7.12}$$

$$\langle\psi_i| = \sum_{j'} U_{j'i} \langle a_{j'}|. \tag{1.7.13}$$

In the new basis,

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{1.7.14}$$

$$= \sum_i p_i \left(\sum_j U_{ij}^\dagger |a_j\rangle \right) \left(\sum_{j'} U_{j'i} \langle a_{j'}| \right) \tag{1.7.15}$$

$$= \sum_{j,j'} \left(\sum_i p_i U_{ij}^\dagger U_{j'i} \right) |a_j\rangle\langle a_{j'}| \tag{1.7.16}$$

$$\equiv \sum_{j,j'} \rho_{jj'} |a_j\rangle\langle a_{j'}|, \tag{1.7.17}$$

where the element $\rho_{jj'}$ is given by

$$\rho_{jj'} = \sum_i U_{j'i} p_i U_{ij}^\dagger \tag{1.7.18}$$

$$= \left(U P U^\dagger \right)^T, \tag{1.7.19}$$

where P is a diagonal matrix whole diagonal elements are p_i. In the new bases $|a_j\rangle$, the off-diagonal element $\rho_{jj'}$ can be nonzero. Indeed, the off-diagonal element $\rho_{jj'}$ is related to the correlation between the two states $|a_j\rangle$ and $|a_{j'}\rangle$.

■ **Example 1.1** Two-Level SystemTL Let's work out an example of an two-basis density matrix. Consider an ensemble of the density matrix

$$\rho = 0|L\rangle\langle L| + 1|R\rangle\langle R|, \tag{1.7.20}$$

or in the matrix form

$$\rho = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.7.21}$$

Keep in mind that the matrix in Eq. (1.7.21) is written in the definitions

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv |L\rangle, \tag{1.7.22}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv |R\rangle. \tag{1.7.23}$$

Now we consider new bases $|X\rangle$ and $|Y\rangle$ (linear polarized states)

$$|L\rangle = \frac{1}{\sqrt{2}}(|X\rangle - i|Y\rangle), \tag{1.7.24}$$

$$|R\rangle = \frac{1}{\sqrt{2}}(|X\rangle + i|Y\rangle). \tag{1.7.25}$$

The unitary transformation is

$$\begin{pmatrix} |L\rangle \\ |R\rangle \end{pmatrix} = U^\dagger \begin{pmatrix} |X\rangle \\ |Y\rangle \end{pmatrix}, \tag{1.7.26}$$

where

$$U^\dagger = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}, \tag{1.7.27}$$

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix}. \tag{1.7.28}$$

In the new bases, using Eq. (1.7.19) the density matrix is

$$\hat{\rho} = \left[\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \right]^T \tag{1.7.29}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}}. \end{pmatrix} \tag{1.7.30}$$

Note that the matrix in Eq. (1.7.30) is written in the definitions

$$\begin{pmatrix}1\\0\end{pmatrix} \equiv |X\rangle, \tag{1.7.31}$$

$$\begin{pmatrix}0\\1\end{pmatrix} \equiv |Y\rangle. \tag{1.7.32}$$

The off-diagonal element $\rho_{YX} = \frac{i}{\sqrt{2}}$ reflects the fact that for a right-circular-polarized state $|R\rangle$, the phase difference between $|X\rangle$ and $|Y\rangle$ is $\pi/2$ (the phase factor $\exp\frac{i\pi}{2} = i$). ■

Exercise 1.8 Density MatrixDM Consider an ensemble of the density matrix

$$\rho = \frac{1}{4}|L\rangle\langle L| + \frac{3}{4}|R\rangle\langle R|. \tag{1.7.33}$$

Calculate the density matrix in the bases $|X\rangle$ and $|Y\rangle$. ■

If we measure an observable A on the ensemble, the expectation value is called "ensemble average",

$$\langle A\rangle = \sum_i p_i \langle\psi_i|A|\psi_i\rangle \tag{1.7.34}$$

$$= \sum_{i,j} p_i \langle\psi_i|\psi_j\rangle\langle\psi_j|A|\psi_i\rangle \tag{1.7.35}$$

$$= \sum_j \langle\psi_j|\underbrace{A\sum_i |\psi_i\rangle p_i\langle\psi_i|}_{\text{This term is } A\rho}|\psi_j\rangle \tag{1.7.36}$$

$$= \sum_j \langle\psi_j|A\rho|\psi_j\rangle \tag{1.7.37}$$

$$= \mathrm{Tr}\,(A\rho). \tag{1.7.38}$$

Although we derive the ensemble average Eq. (1.7.38) in the $|\psi_i$ bases, the trace of a matrix is independent of the bases. Thus, Eq. (1.7.38) is valid in any basis. This basis-free property is the big advantage of using a trace. One direct application is when $A = \mathbb{1}$,

$$\mathrm{Tr}\,(\rho) = \sum_i p_i = 1, \tag{1.7.39}$$

which tells the trace of a density matrix is the total probability.

Exercise 1.9 Properties of Density MatrixPDM Some important properties of density matrix are listed below:

$$\rho = \rho^\dagger$$

(a) $\mathrm{Tr}\,(\rho) = 1$

(c) $0 < \mathrm{Tr}\left(\rho^2\right) \leq 1$

Prove that the above properties are true in any set of bases. ■

Note 1.4 Pure and Mixed Ensemble We start with the bases $|\psi_i\rangle$, where ρ is diagonal. A pure ensemble is specified by $p_i = 1$ of the $|\psi_i\rangle$ and all other $p_j = 0$ for $j \neq i$. The equivalent condition of a pure ensemble is

$$\mathrm{Tr}\left(\rho^2\right) = 1, \tag{1.7.40}$$

which applies to a density matrix in any basis. The condition of a mixed ensemble is

$$\mathrm{Tr}\left(\rho^2\right) < 1. \tag{1.7.41}$$

One particle state is always a pure ensemble. One common mistake is to be confused by the superposition of one particle and the mixed ensemble.[3] Consider a one-particle state (qubit) composed of the superposition of $|0\rangle$ and $|1\rangle$.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle. \tag{1.7.42}$$

One might think that this state has a density matrix $\rho = |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|$. **But, this is wrong!** The correct density matrix is

$$\begin{aligned}
\rho &= |\psi\rangle\langle\psi| && (1.7.43)\\
&= (\alpha|0\rangle + \beta|1\rangle)(\alpha^*\langle 0| + \beta^*\langle 1|) && (1.7.44)\\
&= |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1| + \alpha\beta^*|0\rangle\langle 1| + \alpha^*\beta|1\rangle\langle 0| && (1.7.45)\\
&= \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}, && (1.7.46)
\end{aligned}$$

where

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv |0\rangle, \tag{1.7.47}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv |1\rangle. \tag{1.7.48}$$

It is possible to find the bases where ρ is diagonal, since ρ is a hermitian matrix. The off-diagonal elements in Eq. (1.7.46) describe the correlations between the states $|0\rangle$ and $|1\rangle$.

An example of a mixed ensemble of qubits is

$$\begin{aligned}
\rho &= |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1| && (1.7.49)\\
&= \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}, && (1.7.50)
\end{aligned}$$

where both $|\alpha|^2$ and $|\beta|^2$ are nonzero. In this mixed ensemble, the off-diagonal elements are zero. This means that there is no correlation between the states $|0\rangle$ and $|1\rangle$.

If the number of bases is N, the most random mixed ensemble is

$$\rho_{\mathrm{MR}} = \frac{1}{N}\sum_{i=1}^{N}|\psi_i\rangle\langle\psi_i| \tag{1.7.51}$$

$$= \frac{1}{N}\mathbb{1}_{N\times N}, \tag{1.7.52}$$

where $\mathbb{1}_{N\times N}$ is the N-by-N identity. The off-diagonal elements of the ensemble ρ_{MR} are always zero, i.e., there is not any correlation between the basis states.

Exercise 1.10 Pure Ensemble Which density matrices are pure ensembles?

$$\rho = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$$

(b)

$$\rho = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$$

(c)

$$\rho = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta e^{i\phi} \\ \cos\theta\sin\theta e^{-i\phi} & \sin^2\theta \end{pmatrix}$$

(d)

$$\rho = \begin{pmatrix} \cos^2\theta & \frac{1}{2}\cos\theta\sin\theta \\ \frac{1}{2}\cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$$

■

1.7.2 Dynamics of Density Operators

First, the density operator is not an observable, so we can not use the Heisenberg's picture to obtain its dynamics. Let's begin with a density matrix in the diagonal form,

$$\rho(t) = \sum_i p_i|\psi_i(t)\rangle\langle\psi_i(t)|, \tag{1.7.53}$$

where the dynamics of the states can be obtianed with Schrödinger Picture

$$i\hbar\frac{\partial}{\partial t}|\psi_i(t)\rangle = \mathcal{H}|\psi_i(t)\rangle, \tag{1.7.54}$$

$$-i\hbar\frac{\partial}{\partial t}\langle\psi_i(t)| = \langle\psi_i(t)|\mathcal{H}. \tag{1.7.55}$$

[3] In many places, mixed states are called instead of mixed ensemble, although the latter is properer.

Using Eqs. (1.7.53), (1.7.54) and (1.7.55), we obtain

$$\frac{\partial \rho(t)}{\partial t} = \frac{i}{\hbar}[\rho(t), \mathcal{H}]. \tag{1.7.56}$$

This equation is known as the von Neumann equation or quantum Liouville equation. Equation (1.7.56) describes a closed system where $\mathrm{Tr}\left(\rho^2\right)$ is a constant in time. This means that the coherence of the system is not changed. How could a system have dissipation and decoherence? When a system is open to the environment, the interaction between the system and the environment leads to dissipation and decoherence. The idea is to write $\mathcal{H} = \mathcal{H}_{\mathrm{sys}} + \mathcal{H}_{\mathrm{env}}$ and to derive a equation only about the reduced density matrix

$$\frac{\partial \rho(t)_{\mathrm{sys}}}{\partial t} = \frac{i}{\hbar}\left[\rho(t)_{\mathrm{sys}}, \mathcal{H}_{\mathrm{sys}}\right] + \text{environment terms}, \tag{1.7.57}$$

where the reduced density matrix is obtained by the partial trace

$$\rho(t)_{\mathrm{sys}} = \mathrm{Tr}_{\mathrm{env}}(\rho(t)). \tag{1.7.58}$$

There is not a unique answer how to write the environment terms since that depends on what kind of environment it is and the interaction. The discussions of the environment terms belong to the subject "Open Quantum Systems", which is not the main interest here. We will adopt the phenomenological methods later.

Exercise 1.11 Quantum Liouville Equation Derive the von Neumann equation, Eq. (1.7.56). The first step is to differentiate Eq. (1.7.53). ■

■ **Example 1.2** Dynamics of a Two Level System Let the unperturbed Hamiltonian be

$$\mathcal{H} = \begin{pmatrix} E_c & 0 \\ 0 & E_v \end{pmatrix}, \tag{1.7.59}$$

and write the density matrix in this basis

$$\rho = \begin{pmatrix} \rho_{cc} & \rho_{cv} \\ \rho_{vc} & \rho_{vv} \end{pmatrix}. \tag{1.7.60}$$

Using the von Neumann equation, Eq. (1.7.56), we can obtain four first-order differential equations. Two of them are redundant because $\rho_{cc} + \rho_{vv} = 1$ and $\rho_{cv} = \rho_{vc}^*$. We need only two equations

$$\frac{\partial}{\partial t}\rho_{cc} = 0, \tag{1.7.61}$$

$$\frac{\partial}{\partial t}\rho_{cv} = \frac{i}{\hbar}\rho_{cv}(E_v - E_c), \tag{1.7.62}$$

with the solutions

$$\rho_{cc}(t) = \rho_{cc}(0), \tag{1.7.63}$$

$$\rho_{vv}(t) = \rho_{vv}(0), \tag{1.7.64}$$

$$\rho_{cv}(t) = \rho_{cv}(0)e^{-i\omega_{cv}t}, \tag{1.7.65}$$

$$\tag{1.7.66}$$

with $\omega_{cv} = \frac{E_c - E_v}{\hbar}$. The populations ρ_{cc} and ρ_{vv} are unchanged in a unperturbed system. The off-diagonal element ρ_{cv} has a constant amplitude and a linearly-growing phase in time. This means that the coherence of the system is unchanged. In a realistic situation, the system will be dephased. A phenomenological way to add the dephasing is to add $-\gamma\rho_{cv}$ in Eq. (1.7.62),

$$\frac{\partial}{\partial t}\rho_{cv} = \frac{i}{\hbar}\rho_{cv}\left(E_v - E_c\right) - \gamma\rho_{cv}, \tag{1.7.67}$$

with the solution

$$\rho_{cv}(t) = \rho_{cv}(0)e^{-i\omega_{cv}t-\gamma t}, \tag{1.7.68}$$
$$\tag{1.7.69}$$

and γ is called the dephasing rate. ■

Exercise 1.12 Quantum Liouville Equation Derive Eqs. (1.7.61) and (1.7.62). ■

Bibliography

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2. Quantization of Fields

The strategy to quantize fields is essentially the same as that for a harmonic oscillator. We think electromagnetic modes as some sorts of oscillations. Every mode with a specific frequency ω behaves as a harmonic oscillator. The quantization of a harmonic oscillator is to make $[x,p] = i\hbar$. Here, x and p are canonical variables of the system. The canonical momentum p is a time-derivative of x. In terms of the creation and annihilation operators, we have the relations

$$x \sim a + a^\dagger, \tag{2.0.1}$$
$$p \sim -a + a^\dagger. \tag{2.0.2}$$

The Maxwell's equations read

$$\nabla \cdot (\epsilon(\mathbf{r})\mathbf{E}) = 0 \tag{2.0.3}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{2.0.4}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.0.5}$$
$$\nabla \times \mathbf{B} = \mu(\mathbf{r})\epsilon(\mathbf{r})\frac{\partial \mathbf{E}}{\partial t} \tag{2.0.6}$$

2.1 Single Mode

For an electromagnetic mode of a frequency ω, we look for real solutions of the forms,

$$\mathbf{E}_\omega(\mathbf{r},t) = \boldsymbol{\mathcal{E}}_\omega(\mathbf{r})e^{-i\omega t} + \boldsymbol{\mathcal{E}}^*_\omega(\mathbf{r})e^{i\omega t} \tag{2.1.1}$$
$$\mathbf{B}_\omega(\mathbf{r},t) = \boldsymbol{\mathcal{B}}_\omega(\mathbf{r})e^{-i\omega t} + \boldsymbol{\mathcal{B}}^*_\omega(\mathbf{r})e^{i\omega t}, \tag{2.1.2}$$

which satisfy the Maxwell equations. The solutions to the $\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})$ and $\boldsymbol{\mathcal{B}}_\omega(\mathbf{r})$ will depend on the spatial arrangement of the $\epsilon(\mathbf{r})$ and $\mu(\mathbf{r})$. The filed $\mathbf{E}_{\omega 0}(\mathbf{r})$ satisfies

$$\nabla \cdot (\epsilon(\mathbf{r})\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})) = 0, \tag{2.1.3}$$
$$\nabla \times (\nabla \times \boldsymbol{\mathcal{E}}_\omega(\mathbf{r})) = \mu(\mathbf{r})\epsilon(\mathbf{r})\omega^2\boldsymbol{\mathcal{E}}_\omega(\mathbf{r}). \tag{2.1.4}$$

One can solve the above equations analytically for simple geometries or numerically when geometries are more complicated. Once the $\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})$ is obtained, the magnetic field $\boldsymbol{\mathcal{B}}_\omega(\mathbf{r})$ is given by

$$\begin{aligned} &\nabla \times \boldsymbol{\mathcal{E}}_\omega(\mathbf{r}) = i\omega \boldsymbol{\mathcal{B}}_\omega(\mathbf{r}) \\ &\Rightarrow \boldsymbol{\mathcal{B}}_\omega(\mathbf{r}) = \frac{\nabla \times \boldsymbol{\mathcal{E}}_\omega(\mathbf{r})}{i\omega}. \end{aligned} \tag{2.1.5}$$

The total energy of the mode is

$$\mathcal{H}_\omega = \int dv \left(\frac{\epsilon(\mathbf{r}) E_\omega^2(\mathbf{r})}{2} + \frac{B_\omega^2(\mathbf{r})}{2\mu(\mathbf{r})} \right), \tag{2.1.6}$$

which is similar to

$$\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}. \tag{2.1.7}$$

with the analogies

$$x \sim \mathbf{E}_\omega(\mathbf{r}), \tag{2.1.8}$$

$$p \sim \mathbf{B}_\omega(\mathbf{r}). \tag{2.1.9}$$

It is naturally to speculate[1] that

$$\mathbf{E}_\omega(\mathbf{r}) \sim \boldsymbol{\mathcal{E}}_\omega(\mathbf{r}) a + \boldsymbol{\mathcal{E}}_\omega^*(\mathbf{r}) a^\dagger, \tag{2.1.10}$$

$$\mathbf{B}_\omega(\mathbf{r}) \sim -\boldsymbol{\mathcal{B}}_\omega(\mathbf{r}) a + \boldsymbol{\mathcal{B}}_\omega^*(\mathbf{r}) a^\dagger. \tag{2.1.11}$$

We define the following field operators

$$\mathbf{E}_\omega(\mathbf{r}) = \frac{\left[\boldsymbol{\mathcal{E}}_\omega(\mathbf{r}) a + \boldsymbol{\mathcal{E}}_\omega^*(\mathbf{r}) a^\dagger \right]}{2}, \tag{2.1.12}$$

$$\mathbf{B}_\omega(\mathbf{r}) = \frac{i\left[-\boldsymbol{\mathcal{B}}_\omega(\mathbf{r}) a + \boldsymbol{\mathcal{B}}_\omega^*(\mathbf{r}) a^\dagger \right]}{2} \tag{2.1.13}$$

with the normalization conditions

$$\int dv \epsilon |\mathcal{E}_\omega(\mathbf{r})|^2 = \hbar\omega. \tag{2.1.14}$$

Plugging Eqs. (2.1.12) and (2.1.13) in Eq. (2.1.6), we obtain the Hamiltonian of a single electromagnetic mode,

$$\mathcal{H}_\omega = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right). \tag{2.1.15}$$

[1] You might have the same questions that I had as a student. What are the origins of using a harmonic model to quantize fields? Why is it valid? Why are **E** and **B** the canonical variables? I should say that at least in my opinion, we can not **derive** physics from the first place. All of these steps are hypotheses which are to be examined by experiments. The validities rely on whether the results can explain the observations. To date, it is still the most consistent theory.

All the observables contains the creation and annihilation operator. We can first solve the dynamics of $a(t)$ and obtain all the dynamics. Using the Heisenberg's picture, the equation reads

$$\frac{\partial a}{\partial t} = \frac{i}{\hbar}[\mathcal{H}, a] \tag{2.1.16}$$

$$= -i\omega a, \tag{2.1.17}$$

which has the solution

$$a(t) = a(0)e^{-i\omega t}. \tag{2.1.18}$$

The operator $a^\dagger(t)$ is the hermitian conjugate of $a(t)$,

$$a^\dagger(t) = a^\dagger(0)e^{i\omega t}. \tag{2.1.19}$$

Derivation 2.1 — Bonus Credits!. It requires some efforts to derive Eq. (2.1.15). We sketch the steps

(a) Plug Eqs. (2.1.12) and (2.1.13) in Eq. (2.1.6).

(b) Show that the integral of the magnetic term is equivalent to the electric term. Replace the magnetic term with Eq. (2.1.5). Calculate the integrals with two curls by the integration by parts. Use the identity of vector calculus

$$\int_{\mathcal{V}} dv\mathbf{F}\cdot(\nabla\times\mathbf{A}) = \int_{\mathcal{V}} dv\mathbf{A}\cdot(\nabla\times\mathbf{F}) + \int_{\mathcal{S}} (\mathbf{A}\times\mathbf{F})\cdot d\mathbf{a}, \tag{2.1.20}$$

where $\mathbf{A}$ and $\mathbf{F}$ are arbitrary vector fields. Use Eq. (2.1.4) to get rid of the curls.

(c) Use the normalization condition Eq. (2.1.14).

Note 2.1 Quantization fo Fileds The procedures to quantize a field are:
Find the the eigenmodes (normal modes).
(b) Find the canonical variables.
(c) Define the creation and annihilation operators.
(d) $[a, a^\dagger] = 1$

2.2 Multimode

We have shown how to quantize a single mode of light. We can extend the formulation to multimodes. Let m denote the quantum number of a mode. The total Hamiltonian is

$$\mathcal{H} = \sum_m \hbar\omega_m \left(a_m^\dagger a_m + \frac{1}{2}\right). \tag{2.2.1}$$

For example, m can denote the discrete quantum number of a waveguide, or the continuous quantum number $\mathbf{k}$ of a plane wave. If m are discrete numbers, we have the relations

$$[a_m, a_{m'}^\dagger] = \delta_{mm'}. \tag{2.2.2}$$

The total field is

$$\mathbf{E}(\mathbf{r}) = \sum_m \mathbf{E}_m(\mathbf{r}). \tag{2.2.3}$$

The field operators of the mode m are

$$\mathbf{E}_m(\mathbf{r}) = \frac{\left[\boldsymbol{\mathcal{E}}_m(\mathbf{r})a + \boldsymbol{\mathcal{E}}_m^*(\mathbf{r})a^\dagger\right]}{2}, \tag{2.2.4}$$

$$\mathbf{B}_m(\mathbf{r}) = \frac{i\left[-\boldsymbol{\mathcal{B}}_m(\mathbf{r})a + \boldsymbol{\mathcal{B}}_m^*(\mathbf{r})a^\dagger\right]}{2} \tag{2.2.5}$$

with the normalization conditions

$$\int dv\epsilon|\mathcal{E}_m(\mathbf{r})|^2 = \hbar\omega_m. \tag{2.2.6}$$

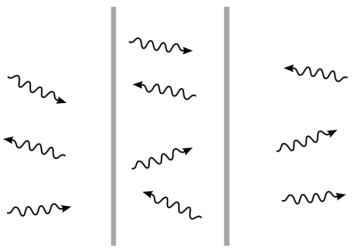
The magnetic field operator is given by

$$\boldsymbol{\mathcal{B}}_m(\mathbf{r}) = \frac{\nabla \times \boldsymbol{\mathcal{E}}_m(\mathbf{r})}{i\omega_m}. \tag{2.2.7}$$

■ **Example 2.1 — Casimir Force in a Nutshell!.** The vacuum energy of the total Hamiltonian is

$$\left\langle 0\left|\sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}}\left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2}\right)\right|0\right\rangle = \sum_{\mathbf{k}} \frac{\hbar\omega_{\mathbf{k}}}{2}. \tag{2.2.8}$$

The integral depends on how many modes there are. The most famous example is the Casimir effect. Consider two parallel metal plates.



The modes in the middle have the wave vector

$$\mathbf{k} = \left(\frac{N\pi}{d}, k_y, k_z\right). \tag{2.2.9}$$

Therefore, the vacuum energy of the middle space is

$$E_0(d) = \frac{\hbar}{2} \times 2 \times \left(\int \frac{L_y dk_y}{2\pi} \int \frac{L_z dk_z}{2\pi}\right) \sum_N c\sqrt{k_y^2 + k_z^2 + \frac{N^2\pi^2}{d^2}}. \tag{2.2.10}$$

This integral is divergent for any separation d. The potential energy of the system $U(d)$ is defined by

$$U(d) = E_0(\infty) - E_0(d). \tag{2.2.11}$$

Although both the two terms are divergent, their difference can be evaluated (See Ref. [**Milonni1992**] or Sec. 2.6 of Ref. [**Gerry2005**]) as

$$U(d) = \frac{-\pi^2 \hbar c L_y L_z}{720 d^3}. \tag{2.2.12}$$

The force per unit area is then

$$\frac{F_c}{L_y L_z} = \frac{1}{L_y L_z}\frac{-\partial U}{\partial d} = -\frac{240\pi^2 \hbar c}{d^4}. \tag{2.2.13}$$

■

2.3 Number States

The eigenstates of the photon Hamiltonian, Eq. (2.2.1) are the direxT product of the number states $|n_1\rangle \otimes |n_2\rangle$.... which is denoted as $|n_1 n_2 ...\rangle$. The total energy of the number states $|n_1 n_2 ...\rangle$ is

$$\langle ...n_2 n_1 | \mathcal{H} | n_1 n_2 ...\rangle = \sum_m \left\langle ...n_2 n_1 \left| \hbar\omega_m \left(a_m^\dagger a_m + \frac{1}{2} \right) \right| n_1 n_2 ... \right\rangle \tag{2.3.1}$$

$$= \sum_m \left(n_m + \frac{1}{2} \right) \hbar\omega_m. \tag{2.3.2}$$

For simplicity, we consider a single-mode system in the following. Since the number states are the eigenstates. The expectation values of the observables are static. The expectation values of $\mathbf{E}(t)$ is

$$\langle \mathbf{E}(t) \rangle = \left\langle n \left| \frac{\left[\boldsymbol{\mathcal{E}}_\omega(\mathbf{r}) a + \boldsymbol{\mathcal{E}}_\omega^*(\mathbf{r}) a^\dagger \right]}{2} \right| n \right\rangle = 0. \tag{2.3.3}$$

The standard deviation of $\mathbf{E}(t)$ of a number state $|n\rangle$ does not vanish

$$\sigma(\mathbf{E}(t)) = \sqrt{\langle \mathbf{E}(t)^2 \rangle - \langle \mathbf{E}(t) \rangle^2} \tag{2.3.4}$$

$$= \sqrt{\langle \mathbf{E}(t)^2 \rangle} \tag{2.3.5}$$

$$= |\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})| \sqrt{\frac{n + \frac{1}{2}}{2}} \tag{2.3.6}$$

Exercise 2.1 — Standard Deviation. Show Eq. (2.3.6). Hint: the operator $\mathbf{E}(t)^2$ is

$$\mathbf{E}(t)^2 = \left(\frac{\left[\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})a + \boldsymbol{\mathcal{E}}^*_\omega(\mathbf{r})a^\dagger\right]}{2}\right)^2 \tag{2.3.7}$$

$$= \frac{|\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})|^2(aa^\dagger + a^\dagger a) + \left[\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})\cdot\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})a^2 + \boldsymbol{\mathcal{E}}^*_\omega(\mathbf{r})\cdot\boldsymbol{\mathcal{E}}^*_\omega(\mathbf{r})(a^\dagger)^2\right]}{4}. \tag{2.3.8}$$

The expectation of $\mathbf{E}(t)^2$ of a number state is

$$\left\langle n\middle|\mathbf{E}(t)^2\middle|n\right\rangle. \tag{2.3.9}$$

■

2.4 Plane Waves

The eigenmodes in vacuum are the plane waves with the quantum number $\mathbf{k}$ and s (polarizations). The eigenmode $\boldsymbol{\mathcal{E}}_m(\mathbf{r})$ is

$$\boldsymbol{\mathcal{E}}_m(\mathbf{r}) = \boldsymbol{\mathcal{E}}_{\mathbf{k},s}(\mathbf{r}) \tag{2.4.1}$$

$$= \frac{1}{\sqrt{V}}\boldsymbol{\mathcal{E}}_{\mathbf{k},s}e^{i\mathbf{k}\cdot\mathbf{r}} \tag{2.4.2}$$

$$= \sqrt{\frac{\hbar\omega}{\epsilon_0 V}}\mathbf{e}_{\mathbf{k},s}e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{2.4.3}$$

where V is the volume where the waves exist. $\mathbf{e}_{\mathbf{k},s}$ denotes the two possible polarizations. The total Hamiltonian reads

$$\mathcal{H} = \sum_{\mathbf{k},s}\hbar\omega_{\mathbf{k}}\left(a^\dagger_{\mathbf{k},s}a_{\mathbf{k},s} + \frac{1}{2}\right). \tag{2.4.4}$$

The electric and magnetic field operators are

$$\mathbf{E}_{\mathbf{k},s}(\mathbf{r}) = \frac{\left[\boldsymbol{\mathcal{E}}_{\mathbf{k},s}a + \boldsymbol{\mathcal{E}}^*_{\mathbf{k},s}(\mathbf{r})a^\dagger\right]}{2}$$

$$= \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\epsilon_0 V}}\frac{\left[\mathbf{e}_{\mathbf{k},s}e^{i\mathbf{k}\cdot\mathbf{r}}a + \mathbf{e}^*_{\mathbf{k},s}e^{-i\mathbf{k}\cdot\mathbf{r}}a^\dagger\right]}{2}, \tag{2.4.5}$$

$$\mathbf{B}_{\mathbf{k},s}(\mathbf{r}) = \frac{\hat{k}}{c}\times\mathbf{E}_{\mathbf{k},s}$$

$$= \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\epsilon_0 V}}\frac{\left[\hat{k}\times\mathbf{e}_{\mathbf{k},s}e^{i\mathbf{k}\cdot\mathbf{r}}a + \hat{k}\times\mathbf{e}^*_{\mathbf{k},s}e^{-i\mathbf{k}\cdot\mathbf{r}}a^\dagger\right]}{2c}. \tag{2.4.6}$$

2.5 Thermal Ensemble

An ensemble of photons is specified by the density matrices. The most classic example is a system in the thermal equilibrium. The equilibrium is reached when a photonic

system is in contact with a heat reservoir (environment). For a given temperature T, according to statistical mechanics, the probability to occupy a state n is proportional to

$$p(n) \sim e^{-\frac{E_n}{k_B T}}, \tag{2.5.1}$$

where k_B is the Boltzmann's constant. Considering the normalization, the probability is

$$p(n) = \frac{e^{-\frac{E_n}{k_B T}}}{\sum_m e^{-\frac{E_m}{k_B T}}} \tag{2.5.2}$$

$$= \frac{e^{-\frac{E_n}{k_B T}}}{Z}, \tag{2.5.3}$$

with the partition function Z

$$Z = \sum_m e^{-\frac{E_m}{k_B T}}. \tag{2.5.4}$$

Thus, the density operator of a thermal ensemble is

$$\rho_{\text{th}} = \sum_n p(n)|n\rangle\langle n| \tag{2.5.5}$$

$$= \frac{\sum_n e^{-\frac{E_n}{k_B T}}|n\rangle\langle n|}{Z} \tag{2.5.6}$$

$$= \frac{e^{-\frac{\mathcal{H}}{k_B T}}}{\text{Tr}[e^{-\frac{\mathcal{H}}{k_B T}}]} \tag{2.5.7}$$

Exercise 2.2 — Partition Function. Show that the partition function Z of a single mode photonic system is

$$Z = \frac{\exp\left(-\frac{\hbar\omega}{2k_B T}\right)}{1 - \exp\left(-\frac{\hbar\omega}{k_B T}\right)}. \tag{2.5.8}$$

Use $E_m = \left(m + \frac{1}{2}\right)\hbar\omega$ in Eq. (2.5.4) ■

The average number of the thermal ensemble is

$$\begin{aligned}
\langle \hat{N} \rangle &= \mathrm{Tr}[\rho_{\mathrm{th}} \hat{N}] && (2.5.9)\\
&= \sum_m \langle m | \rho_{\mathrm{th}} \hat{N} | m \rangle && (2.5.10)\\
&= \sum_m m \langle m | \rho_{\mathrm{th}} | m \rangle && (2.5.11)\\
&= \sum_{m,n} \frac{m e^{-\frac{\hbar\omega(n+1/2)}{k_B T}}}{Z} \langle m|n\rangle\langle n|m\rangle && (2.5.12)\\
&= \sum_m \frac{m e^{-\frac{\hbar\omega(m+1/2)}{k_B T}}}{Z} \quad \text{See Derivation 2.3} && (2.5.13)\\
&= \frac{1}{\exp \frac{\hbar\omega}{k_B T} - 1}, && (2.5.14)
\end{aligned}$$

which is the Bose-Einstein distribution.

Derivation 2.2 Trick of Sums of Series trick Let

$$\tilde{Z}(x) = \sum_{m=0}^{\infty} e^{-mx} = \frac{1}{1 - e^{-x}}. \tag{2.5.15}$$

The trick to calculate the following sums

$$\tilde{Z}_l(x) \equiv \sum_{m=0}^{\infty} m^l e^{-mx}, \tag{2.5.16}$$

where l is an integer, is from the relation

$$\tilde{Z}_l(x) = (-1)^l \frac{\partial^l \tilde{Z}}{\partial x^l}. \tag{2.5.17}$$

Exercise 2.3 — Standard Derivation of $\hat{N}$. Calculate $\sigma\left(\hat{N}\right)$ of an thermal ensemble of temperature T. Use

$$\begin{aligned}
\sigma\left(\hat{N}\right) &= \sqrt{\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2}, && (2.5.18)\\
\langle \hat{N} \rangle &= \mathrm{Tr}[\rho_{\mathrm{th}} \hat{N}], && (2.5.19)\\
\langle \hat{N}^2 \rangle &= \mathrm{Tr}[\rho_{\mathrm{th}} \hat{N}^2]. && (2.5.20)
\end{aligned}$$

■

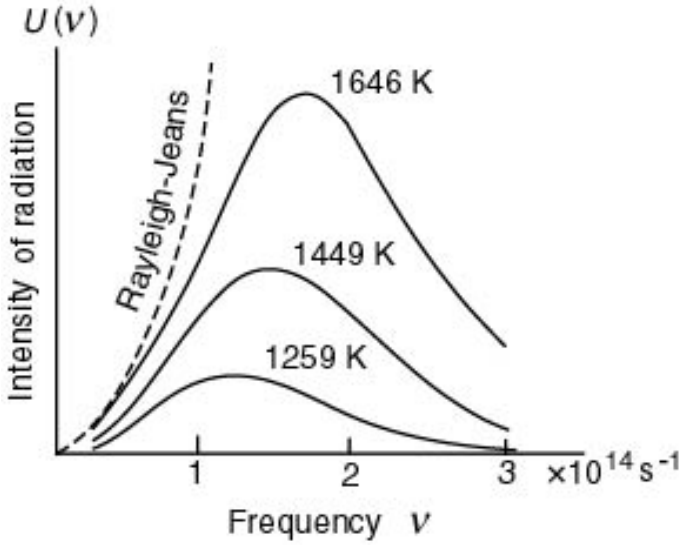


Figure 2.1: Energy density of a thermal ensemble of photons. Image Source

2.6 Black-Body Radiation

The average energy of one single mode is $\langle \hat{N} \rangle \hbar\omega$. The density of state of a frequency per unit volume $g(\omega)$ is

$$g(\omega) = \frac{\omega^2}{\pi^2 c^3}. \tag{2.6.1}$$

The average energy density $U(\omega)$ is

$$U(\omega) = \langle \hat{N} \rangle \hbar\omega g(\omega) \tag{2.6.2}$$

$$= \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{\exp \frac{\hbar\omega}{k_B T} - 1}. \tag{2.6.3}$$

Its classical analog is the Rayleigh-Jeans formula

$$U(\omega) = g(\omega) k_B T = \frac{\omega^2}{\pi^2 c^3} k_B T, \tag{2.6.4}$$

which leads to the ultraviolet catastrophe of the classical physics.

Derivation 2.3 — Density of States. A cuboid has the side lengths L_x, L_y and L_z. The allowed wave vectors are

$$k_x = \frac{2\pi l_x}{L_x} \tag{2.6.5}$$

$$k_y = \frac{2\pi l_y}{L_y} \tag{2.6.6}$$

$$k_z = \frac{2\pi l_z}{L_z} \tag{2.6.7}$$

where l_x, l_y and l_z are integers. The change of the total number m of modes is

$$\Delta m = 2\Delta l_x \Delta l_y \Delta l_z = 2\left(\frac{L_x L_y L_z}{(2\pi)^3}\right)\Delta k_x \Delta k_y \Delta k_z, \tag{2.6.8}$$

where the factor 2 accounts for the polarizations. In the continuum limit, it becomes

$$\frac{dm}{V} = \left(\frac{1}{4\pi^3}\right) dk_x dk_y dk_z \tag{2.6.9}$$

$$= \frac{1}{4\pi^3} 4\pi k^2 dk \tag{2.6.10}$$

$$= \frac{1}{\pi^2}\frac{\omega^2 d\omega}{c^3}, \tag{2.6.11}$$

$$\Rightarrow g(\omega) \equiv \frac{1}{V}\frac{dm}{d\omega} = \frac{\omega^2}{\pi^2 c^3}. \tag{2.6.12}$$

2.7 Quadrature Operators

We have applied the ideas of a harmonic oscillator to quantize fields. The conjugate variables of a particle are x and p, which are just numbers. Unlike a particle, an photon has field operators $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$, which have values at every position. The similarities of them are the creation and annihilation operators a and $a^\dagger$. It is then useful to define the **dimensionless** operators for photons. We introduce the quadrature operators,

$$X = \frac{a + a^\dagger}{2}, \tag{2.7.1}$$

$$Y = \frac{a - a^\dagger}{2i}. \tag{2.7.2}$$

The operator X is the dimensionless position operator, and the operator Y is the dimensionless momentum. They have the relation

$$[X, Y] = \frac{i}{2}. \tag{2.7.3}$$

Using the generalized uncertainty relation, we obtain

$$\sigma(X)\sigma(Y) \geq \frac{|\langle [X, Y]\rangle|}{2} = \frac{1}{4}. \tag{2.7.4}$$

The electric field operator of a mode m is rewritten as

$$\mathbf{E}_m(\mathbf{r}) = \mathrm{Re}[\boldsymbol{\mathcal{E}}_m(\mathbf{r})]X - \mathrm{Im}[\boldsymbol{\mathcal{E}}_m(\mathbf{r})]Y. \tag{2.7.5}$$

In the case of plane waves, the electric field operator of a mode $\{\mathbf{k}, s\}$ is

$$\mathbf{E}_{\mathbf{k},s}(\mathbf{r}) = \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\epsilon_0 V}}\left\{\mathrm{Re}[\mathbf{e}_{\mathbf{k},s}]\cos(\mathbf{k}\cdot\mathbf{r})X - \mathrm{Im}[\mathbf{e}^*_{\mathbf{k},s}]\sin(\mathbf{k}\cdot\mathbf{r})Y\right\}. \tag{2.7.6}$$

Bibliography

[1] P. W. Milonni and M.-L. Shih, *Contemporary Physics*, volume 33, number 5, pages 313-322, 1992

[2] C. Gerry and P. Knight, *Introductory Quantum Optics*, Cambridge University Press, 2005

3. Phase Space Pictures

The state of a classical particle is fully determined by its x and p. A useful way to represent the states is the phase space (x, p), where the horizontal axis is x and the vertical axis is p. A state of a classical particle is one point in the phase phase. The time evolution of a state is the trajectory traveled by the particle. The classic example is the harmonic oscillator with

$$x(t) = x_0 \cos(\omega t + \phi), \tag{3.0.1}$$

$$p(t) = -\omega x_0 \sin(\omega t + \phi), \tag{3.0.2}$$

or in the dimensionless expression

$$\tilde{x}(t) = \frac{x(t)}{x_0} = \cos(\omega t + \phi), \tag{3.0.3}$$

$$\tilde{p}(t) = \frac{p(t)}{\omega x_0} = -\sin(\omega t + \phi). \tag{3.0.4}$$

The state travels along the trajectory is a unit circle (see Fig. 3.1).

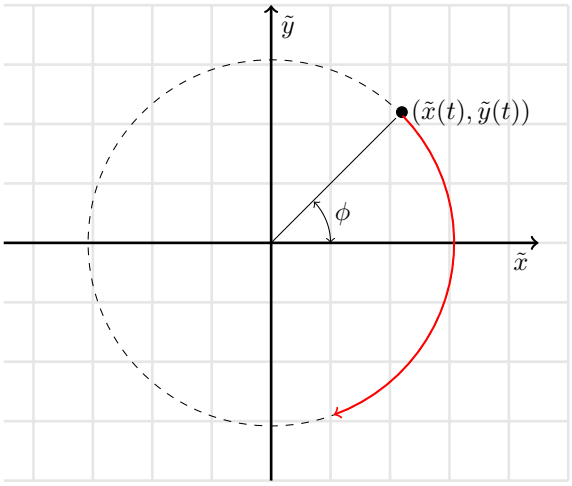


Figure 3.1: A classical state is a point in the phase space. The motion of a state is a trajectory. In the case of a harmonic ocsillator, the trjectory is a circle.

An ensemble of classical particles are described by the phase space probability density $f(x,p)$, where to find a particle with a position x and a momentum p is given by

$$f(x,p)dxdp, \tag{3.0.5}$$

and the normalization condition is

$$\int dx \int dp f(x,p)dxdp = 1. \tag{3.0.6}$$

Classically, the function $f(x,p)$ of a pure state, i.e., a single particle, is a delta function $f(x,p) = \delta(x-x_0)\delta(p-p_0)$.

We have make the analogies $x \leftrightarrow X$ and $y \leftrightarrow Y$. One question arises: can we define a function similar to $f(x,p)$ to describe states or ensembles of photons? The problem is that a quantum state can not have well-defined X and Y at the same time. Thus, a quantum state is not a single point in the phase space. For a coherent state $|\alpha\rangle$, we have the relations

$$X = \frac{\alpha + \alpha^*}{2}, \tag{3.0.7}$$

$$Y = \frac{\alpha - \alpha^*}{2i}. \tag{3.0.8}$$

The state is a blurred circle in the phase space (see Fig. 3.2) because of the uncertainty relations. Coherent states are the states satisfying the minimum uncertainty relations. In general, an arbitrary state can have a very broad distribution in the phase space.

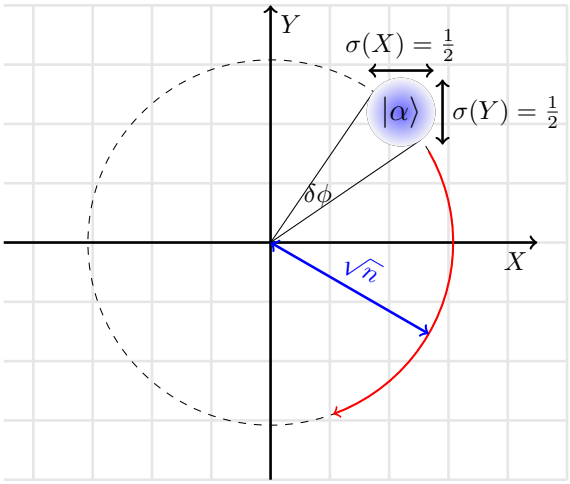


Figure 3.2: A coherent state is a fuzzy circle in the phase space.

Quantum states can be represented in the (X,Y) space, or equivalently the complex α space. It is then to desire to define a probability density $f(X,Y)$ or $f(\alpha)$. However, there is not a unique way to define a probability density of a quantum state. We are going to introduce the three most often distributions,

- Wigner distribution
- Q-function
- P-function

Note that the definitions and calculations of these functions are quite mathematically involved. These functions serve as quantitative tools to describe the phase space probability densities. It is fine to have a qualitative picture in mind first and know more calculations when it is needed.

3.1 Properties of Coherent States

3.1.1 Orthogonality

Two coherent states $|\alpha\rangle$ and $|\beta\rangle$ are not orthogonal,

$$\langle\beta|\alpha\rangle = e^{-\frac{(|\alpha|^2+|\beta|^2)}{2}} \sum_{n,m} \frac{(\beta^*)^m(\alpha)^n}{\sqrt{m!n!}}\langle m|n\rangle \tag{3.1.1}$$

$$= e^{-\frac{(|\alpha|^2+|\beta|^2)}{2}} \sum_{n} \frac{(\beta^*)^n(\alpha)^n}{n!} \tag{3.1.2}$$

$$= e^{-\frac{(|\alpha|^2+|\beta|^2)}{2}} e^{\beta^*\alpha} \tag{3.1.3}$$

$$= e^{-\frac{|\alpha-\beta|^2}{2}} e^{\frac{\beta^*\alpha-\beta\alpha^*}{2}}, \tag{3.1.4}$$

which does not vanish.

3.1.2 Identity

The identity can be expressed with the coherent states,

$$\int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha| \equiv \int \frac{d\mathrm{Re}[\alpha]d\mathrm{Im}[\alpha]}{\pi}|\alpha\rangle\langle\alpha| = \mathbb{1}. \tag{3.1.5}$$

Derivation 3.1 — Identity with Coherent States. The proof of Eq. (3.1.5) is as follows. Let $\alpha = re^{\phi}$ and.

$$d\alpha^2 = d\mathrm{Re}[\alpha]d\mathrm{Im}[\alpha] = rdrd\theta. \tag{3.1.6}$$

The left hand side of Eq. (3.1.5) becomes

$$\int \frac{rdrd\theta}{\pi}|\alpha\rangle\langle\alpha| = \int \frac{rdrd\theta}{\pi} e^{-r^2} \sum_{m,n} \frac{e^{i(n-m)\theta} r^{m+n}}{n!}|m\rangle\langle n| \tag{3.1.7}$$

$$= \sum_n \frac{\int dr e^{-r^2} 2r^{2n+1}}{n!}|n\rangle\langle n| \tag{3.1.8}$$

$$= \sum_n \frac{\int du e^{-u} u^n}{n!}|n\rangle\langle n| \tag{3.1.9}$$

$$= \sum_n |n\rangle\langle n| = \mathbb{1}. \tag{3.1.10}$$

3.1.3 Coherent State Representations of Operators

Any operator X can be expressed in the coherent state bases with the identity Eq. (3.1.5),

$$X = \int \frac{d^2\alpha}{\pi} \int \frac{d^2\beta}{\pi} |\alpha\rangle\langle\alpha|X|\beta\rangle\langle\beta|. \tag{3.1.11}$$

However, coherent states are not orthogonal, so the coherent states form an **overcomplete** set of bases.[1] It is possible to write X in the coherent state diagonal form.

An operator X is uniquely determined $\langle\alpha|X|\alpha\rangle$. The diagonal element $\langle\alpha|A|\alpha\rangle$ in the number state basis is

$$\langle\alpha|X|\alpha\rangle = \exp -|\alpha|^2 \sum_{m,n} \frac{\langle n|X|m\rangle \alpha^m (\alpha^*)^n}{\sqrt{m!n!}}, \tag{3.1.12}$$

indicating that $\langle\alpha|X|\alpha\rangle$ contains all the information of the elements $\langle n|X|m\rangle$, which forms a complete set.

Coherent state diagonal representation. Suppose that X has a series expansion of a and $a^\dagger$ in the antinormal ordering,

$$X = \sum_{mn} \chi^A_{nm} a^n (a^\dagger)^m, \tag{3.1.13}$$

where X_{nm} is a c-number. The subscript A denotes the antinormal ordering. Inserting the identity, we obtain

$$X = \sum_{mn} \chi^A_{nm} a^n \left(\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| \right) (a^\dagger)^m \tag{3.1.14}$$

$$= \int d^2\alpha \chi^A(\alpha) |\alpha\rangle\langle\alpha| \tag{3.1.15}$$

where

$$\chi^A(\alpha) = \frac{1}{\pi} \sum_{mn} \chi^A_{nm} \alpha^n (\alpha^*)^m, \tag{3.1.16}$$

is a c-number.

3.2 Phase Space Distributions

Given a density matrix ρ, there are three important distribution functions which are the quantum analogs of the classical probability density $f(x,p)$.

3.3 Wigner Distribution

The Wigner function $W(\alpha)$ is defined as

$$W(\alpha) = \int \frac{d^2\eta}{\pi^2} e^{\eta^*\alpha - \eta\alpha^*} \chi_W(\eta), \tag{3.3.1}$$

where the characteristic function $\chi_W(\eta)$ is

$$\chi_W(\eta) = \mathrm{Tr}\left[\rho e^{\eta a^\dagger - \eta^* a}\right]. \tag{3.3.2}$$

[1] See Se. 5.4. of Ref. [**Garrison2008**] for a more rigorous discussion.

Exercise 3.1 — Normalization. Show that

$$\int \frac{d^2\alpha}{\pi^2} e^{\eta^*\alpha - \eta\alpha^*} = \delta_2(\eta) \equiv \delta(\mathrm{Re}[\eta])\delta(\mathrm{Im}[\eta]), \tag{3.3.3}$$

and use the result and Eq. (3.3.1) to show

$$\int d^2\alpha W(\alpha) = 1. \tag{3.3.4}$$

Hint: a delta function can be expressed as

$$\delta(x) = \frac{1}{2\pi}\int e^{ikx} dk. \tag{3.3.5}$$

■

The ensemble average of an operator X in this representation is

$$\langle X\rangle = \int d^2\alpha \chi^W(\alpha) W(\alpha), \tag{3.3.6}$$

where

$$\chi^W(\alpha) = \sum_{n,m} \chi^W_{nm} \alpha^n (\alpha^*)^m \tag{3.3.7}$$

The coefficient χ^W_{nm} is the Weyl(symmetric)-ordering representation of an operator X,

$$X = \sum_{m,n} \chi^W_{nm} \left(\frac{(a^\dagger)^n a^m + a^m (a^\dagger)^n}{2} \right). \tag{3.3.8}$$

3.4 Glauber–Sudarshan P-function

The P-function is defined by

$$\rho = \int d^2\alpha P(\alpha) |\alpha\rangle\langle\alpha|, \tag{3.4.1}$$

and satisfies the normalization condition

$$1 = \mathrm{Tr}[\rho] = \int d^2\alpha P(\alpha). \tag{3.4.2}$$

The P-function can be obtained by the normal-ordering characteristic function

$$P(\alpha) = \int \frac{d^2\eta}{\pi^2} e^{\eta^*\alpha - \eta\alpha^*} \chi_N(\eta), \tag{3.4.3}$$

where the characteristic function $\chi_N(\eta)$ is

$$\chi_N(\eta) = \mathrm{Tr}\left[\rho e^{\eta a^\dagger} e^{-\eta^* a}\right]. \tag{3.4.4}$$

The ensemble average of an operator X in this representation is

$$\langle X\rangle = \int d^2\alpha \chi^N(\alpha)P(\alpha), \tag{3.4.5}$$

where

$$\chi^N(\alpha) = \sum_{n,m} \chi^N_{nm}\alpha^n(\alpha^*)^m \tag{3.4.6}$$

The coefficient χ^N_{nm} is the normal-ordering representation of an operator X,

$$X = \sum_{m,n} \chi^N_{nm}(a^\dagger)^m a^n. \tag{3.4.7}$$

Note 3.1 — Classical States and Nonclassical States. A state with $P(\alpha) < 0$ is defined as a nonclassical state.

3.5 Q-function

The Q-function is defined by

$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle, \tag{3.5.1}$$

which is always positive since it is the diagonal element of the density matrix. The function $Q(\alpha)$ satisfies

$$0 \le Q(\alpha) \le \frac{1}{\pi}, \tag{3.5.2}$$

and

$$\mathrm{Tr}[\rho] = \int d^2\alpha Q(\alpha) = 1. \tag{3.5.3}$$

The Q-function can be obtained by the antinormal-ordering characteristic function

$$Q(\alpha) = \int \frac{d^2\eta}{\pi^2} e^{\eta^*\alpha - \eta\alpha^*}\chi_A(\eta), \tag{3.5.4}$$

where the characteristic function $\chi_A(\eta)$ is

$$\chi_A(\eta) = \mathrm{Tr}\left[\rho e^{-\eta^* a}e^{\eta a^\dagger}\right]. \tag{3.5.5}$$

The ensemble average of an operator X in this representation is

$$\langle X\rangle = \int d^2\alpha \chi^A(\alpha)Q(\alpha), \tag{3.5.6}$$

where

$$\chi^A(\alpha) = \sum_{n,m} \chi^A_{nm}\alpha^n(\alpha^*)^m. \tag{3.5.7}$$

The coefficient χ^A_{nm} is the antinormal-ordering representation of an operator X,

$$X = \sum_{m,n} \chi^A_{nm}a^n(a^\dagger)^m. \tag{3.5.8}$$

	$W(\alpha)$	$Q(\alpha)$	$P(\alpha)$
coherent state $\|\alpha_0\rangle$	$\frac{2}{\pi}e^{-2\|\alpha-\alpha_0\|^2}$	$\frac{1}{\pi}e^{-\|\alpha-\alpha_0\|^2}$	$\delta^2(\alpha-\alpha_0)$
thermal ensemble	$\frac{1}{\pi(\bar{n}+1/2)}\exp\left(-\frac{\|\alpha\|^2}{\bar{n}+1/2}\right)$	$\frac{1}{\pi(\bar{n}+1)}\exp\left(-\frac{\|\alpha\|^2}{\bar{n}+1}\right)$	$\frac{1}{\pi(\bar{n})}\exp\left(-\frac{\|\alpha\|^2}{\bar{n}}\right)$
pure number ensemble $\|1\rangle\langle 1\|$	$-(1-4\|\alpha\|^2)\frac{2}{\pi}e^{-2\|\alpha\|^2}$	$\frac{\|\alpha\|^2}{\pi}e^{-\|\alpha\|^2}$	singular

Table 3.1: Examples of $W(\alpha)$, $Q(\alpha)$, and $P(\alpha)$

Note 3.2 — Coherent States. (a) The phase space of photon states or ensembles are described by the two dimensional complex α plane.

(b) The real part and imaginary part of α are related to the quadrature operator X and Y.

$$X = \mathrm{Re}[\alpha], \tag{3.5.9}$$

$$Y = \mathrm{Im}[\alpha]. \tag{3.5.10}$$

(c) A coherent state $|\alpha_0\rangle$ is a fuzzy circle on the complex α plane.

(d) The coherent states are not orthogonal, so they are overcomplete.

(e) There are three ways to write the probability density

- Wigner distribution $W(\alpha)$: symmetric ordering
- Q-function $Q(\alpha)$: antinormal ordering
- P-function $P(\alpha)$: normal ordering

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4. Light-Matter Interaction

Light-matter interactions occur when charged particles accelerate in time-dependent electric field. An accelerating charge particle generates light, and conversely electric fields cause forces on charges particle. In most scenarios, magnetic field does not directly interact with matter since it is easier to have charges, electric dipoles than magnetic dipoles.

Time-dependent charges can be described by a charge density $\rho(\mathbf{r},t)$. It is more often to use dipoles and currents to describe light-matter interaction. Polarization $\mathbf{P}$ (dipole) and currents density $\mathbf{J}$ have the relations

$$\nabla\cdot\mathbf{J}+\frac{\partial\rho}{\partial t}=0, \tag{4.0.1}$$

$$\mathbf{J}=\frac{\partial\mathbf{P}}{\partial t}. \tag{4.0.2}$$

4.1 Hamiltonian

4.1.1 Interaction Hamiltonian

According to classical mechanics, a charge particle has the Hamiltonian (SI units)

$$\mathcal{H}=\frac{(\mathbf{p}-q\mathbf{A})^2}{2m}+q\Phi(\mathbf{r},t), \tag{4.1.1}$$

where q is the charge of the particle not the position. $\Phi(\mathbf{r},t)$ is the electric potential. In the case of an electron, $q=-e$, we have

$$\mathcal{H}=\frac{(\mathbf{p}+e\mathbf{A})^2}{2m}-e\Phi(\mathbf{r},t). \tag{4.1.2}$$

We can decompose it into $\mathcal{H}_0$ and $\mathcal{H}_I$,

$$\mathcal{H}_0=\frac{p^2}{2m}, \tag{4.1.3}$$

$$\mathcal{H}_I=\frac{e\,(\mathbf{p}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{p})}{2m}+\frac{e^2A^2}{2m}-e\Phi. \tag{4.1.4}$$

Typically, the term $\frac{e^2A^2}{2m}$ is dropped since the momentum of field $e\mathbf{A}$ is usually small than the electron's momentum $\mathbf{p}$ [1]. Since the momentum $\mathbf{p}$ is a differential operator, $\mathbf{p}\cdot\mathbf{A}$ is not equal to $\mathbf{A}\cdot\mathbf{p}$. The vector potential $\mathbf{A}$ and Coulomb's potential Φ are not unique. The Maxwell's equations are invariant under the gauge transformations

$$\mathbf{A}' = \mathbf{A} + \nabla\lambda(\mathbf{r}, t), \tag{4.1.5}$$

$$\Phi' = \Phi - \frac{\partial\lambda(\mathbf{r}, t)}{\partial t}. \tag{4.1.6}$$

The fields are given by

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{4.1.7}$$

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}. \tag{4.1.8}$$

The Gauge $\nabla\cdot\mathbf{A} = 0$ is frequently used in quantum optics. In this gauge, the interaction Hamiltonian (dropping $\frac{e^2A^2}{2m}$) becomes

$$\mathcal{H}_I = \frac{e(\mathbf{A}\cdot\mathbf{p})}{m} - e\Phi. \tag{4.1.9}$$

If $\Phi = 0$ is chosen [2], the interaction Hamiltonian becomes

$$\mathcal{H}_I = \frac{e(\mathbf{A}\cdot\mathbf{p})}{m} \tag{4.1.10}$$

$$= -\int dv\mathbf{A}\cdot\mathbf{J} \tag{4.1.11}$$

where we use $\int dv\mathbf{J} = \frac{-e\mathbf{p}}{m}$. Another choice is the Göppert-Mayer gauge,

$$\lambda = -(\mathbf{r} - \mathbf{r}_0)\cdot\mathbf{A}(\mathbf{r}_0). \tag{4.1.12}$$

In this gauge, we have

$$\mathbf{A}' = \mathbf{A}(\mathbf{r}) - \mathbf{A}(\mathbf{r}_0), \tag{4.1.13}$$

$$\Phi' = -e(\mathbf{r} - \mathbf{r}_0)\cdot\mathbf{E}(\mathbf{r}_0) \equiv -\mathbf{d}\cdot\mathbf{E}, \tag{4.1.14}$$

where the dipole operator is $-e(\mathbf{r}-\mathbf{r}_0)$. The so-called dipole approximation is when $\mathbf{A}(\mathbf{r})$ is almost a constant, i.e., $\mathbf{A}(\mathbf{r}) \simeq \mathbf{A}(\mathbf{r}_0)$. This approximation is valid when the charge distributions are within a small region. The interaction Hamiltonian becomes

$$\mathcal{H}_I = -\mathbf{E}\cdot\mathbf{d} \tag{4.1.15}$$

[1] Well, this is a sloppy argument. In electromagnetism, the higher-order terms of the vector potential$\mathbf{A}$ are relativistic. In this viewpoint, the term $\frac{e^2A^2}{2m}$ is proportional to $\frac{v^2}{c^2}$.

[2] In the region without charges $\nabla\cdot\mathbf{E} = 0$, we can define $\mathbf{E} = -\nabla\Phi$. Using the gauge transformation $\lambda = \int\Phi dt$, we can eliminate Φ and make $\nabla\cdot\mathbf{A} = 0$.

4.1.2 Total Hamiltonian

The total Hamiltonian of the light-matter is

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I + \mathcal{H}_F. \tag{4.1.16}$$

where

$$\mathcal{H}_F = \sum_m \int dv \left(\frac{\epsilon(\mathbf{r}) E_m^2(\mathbf{r})}{2} + \frac{B_m^2(\mathbf{r})}{2\mu(\mathbf{r})} \right). \tag{4.1.17}$$

The Hamiltonian of matter $\mathcal{H}_0$ is not necessary of the form of a free particle. In general, $\mathcal{H}_0$ describes a N-level system,

$$\mathcal{H}_0 = \sum_n E_n |E_n\rangle\langle E_n|. \tag{4.1.18}$$

The simplest case is a two level system (TLS)

$$\mathcal{H}_{TLS} = \begin{pmatrix} E_c & 0 \\ 0 & E_v \end{pmatrix}. \tag{4.1.19}$$

The interaction Hamiltonian for a a two level system is

$$\mathcal{H}_I = \begin{pmatrix} \langle E_c| - \mathbf{E}\cdot\mathbf{d}|E_c\rangle & \langle E_c| - \mathbf{E}\cdot\mathbf{d}|E_v\rangle \\ \langle E_v| - \mathbf{E}\cdot\mathbf{d}|E_c\rangle & \langle E_v| - \mathbf{E}\cdot\mathbf{d}|E_v\rangle \end{pmatrix} \tag{4.1.20}$$

$$= -\mathbf{E}\cdot\begin{pmatrix} \mathbf{d}_{cc} & \mathbf{d}_{cv} \\ \mathbf{d}_{vc} & \mathbf{d}_{vv} \end{pmatrix}, \tag{4.1.21}$$

where the dipole matrix element is $\mathbf{d}_{nn'} = \langle E_n|\mathbf{d}|E_{n'}\rangle$. In many cases, the diagonal elements of dipole matrices vanishes since the eigenfunctions are typical symmetric.

4.2 Classical Fields and Quantum Matter

We consider that the matter is described by a N-level system and treat the electric field $\mathbf{E}(\mathbf{r}, t)$ as a number. The Hamiltonian is

$$\mathcal{H} = \sum_n E_n |E_n\rangle\langle E_n| - \mathbf{E}\cdot\mathbf{d}. \tag{4.2.1}$$

In the case of a TLS system, the Hamiltonian is

$$\mathcal{H} = \begin{pmatrix} E_c & 0 \\ 0 & E_v \end{pmatrix} - \mathbf{E}\cdot\begin{pmatrix} 0 & \mathbf{d}_{cv} \\ \mathbf{d}_{vc} & 0 \end{pmatrix}, \tag{4.2.2}$$

where we assume the diagonal elements of the dipole matrix are zeros. To solve the dynamics, we start with the interaction picture where state is

$$|\psi\rangle = C_c(t) e^{-i\omega_c t}|E_c\rangle + C_v(t) e^{-i\omega_v t}|E_v\rangle. \tag{4.2.3}$$

It is clear that without an external field $\mathbf{E}$, the coefficients $C_c(t)$ and $C_v(t)$ are constant in time. Plugging Eq. (4.2.3) in the Schrödinger equation, we obtain

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}C_c\\C_v\end{pmatrix} = -\mathbf{E}\cdot\begin{pmatrix}0 & \mathbf{d}_{cv}e^{i(\omega_c-\omega_v)t}\\ \mathbf{d}_{vc}e^{i(\omega_v-\omega_c)t} & 0\end{pmatrix}\begin{pmatrix}C_c\\C_v\end{pmatrix}. \tag{4.2.4}$$

The dipole matrix elements in the interaction picture oscillate rapidly in time. The electric field $\mathbf{E} = \boldsymbol{\mathcal{E}}_\omega e^{-i\omega t} + \boldsymbol{\mathcal{E}}^*_\omega e^{i\omega t}$ needs to have a frequency $\omega \simeq (\omega_c - \omega_v)$ in order to create transition. We write

$$\omega = \omega_{cv} + \Delta, \tag{4.2.5}$$

where $\omega_{cv} = \omega_c - \omega_v$ and Δ is the detuning.

4.2.1 Rabi Model

Let the external field $\mathbf{E} = \mathbf{E}_0\cos\omega t = \mathbf{E}_0\left(\frac{e^{-i\omega t}+e^{i\omega t}}{2}\right)$. The equation of the coefficients is

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}C_c\\C_v\end{pmatrix} = \begin{pmatrix}0 & \frac{V_0}{2}\left[e^{-i\Delta t}+e^{i(2\omega_{cv}+\Delta)t}\right]\\ \frac{V_0^*}{2}\left[e^{i\Delta t}+e^{-i(2\omega_{cv}+\Delta)t}\right] & 0\end{pmatrix}\begin{pmatrix}C_c\\C_v\end{pmatrix}. \tag{4.2.6}$$

where

$$V_0 = -\mathbf{E}_0\cdot\mathbf{d}_{cv}. \tag{4.2.7}$$

The equation needs to be solved numerically. The rotating-wave-approximation (RWA), where the high frequency terms are dropped is often used. Under the RWA, the equation reads

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}C_c\\C_v\end{pmatrix} = \begin{pmatrix}0 & \frac{V_0}{2}e^{-i\Delta t}\\ \frac{V_0^*}{2}e^{i\Delta t} & 0\end{pmatrix}\begin{pmatrix}C_c\\C_v\end{pmatrix}. \tag{4.2.8}$$

Eliminating the variable C_v, we obtain the second-order differential equation

$$\ddot{C}_c + i\Delta\dot{C}_c + \frac{|V_0|^2}{4\hbar^2}C_c = 0. \tag{4.2.9}$$

The general solution is

$$C_c(t) = A_+e^{i\lambda_+ t} + A_-e^{i\lambda_- t} \tag{4.2.10}$$

with

$$\lambda_\pm = \Delta \pm \sqrt{\Delta^2 + \frac{|V_0|^2}{\hbar^2}} \equiv \Delta \pm \Omega_R. \tag{4.2.11}$$

The Rabi frequency $\Omega_R = \sqrt{\Delta^2 + \frac{|V_0|^2}{\hbar^2}}$. If initially $C_v(0) = 1$, the solution is

$$C_e = e^{i\frac{\Delta t}{2}}\frac{iV_0}{\hbar\Omega_R}\sin\frac{\Omega_R t}{2}, \tag{4.2.12}$$

$$C_g = e^{i\frac{\Delta t}{2}}\left[\cos\frac{\Omega_R t}{2} - i\frac{\Delta}{\Omega_R}\sin\frac{\Omega_R t}{2}\right]. \tag{4.2.13}$$

It can be checked that $|C_c|^2 + |C_v|^2 = 1$.

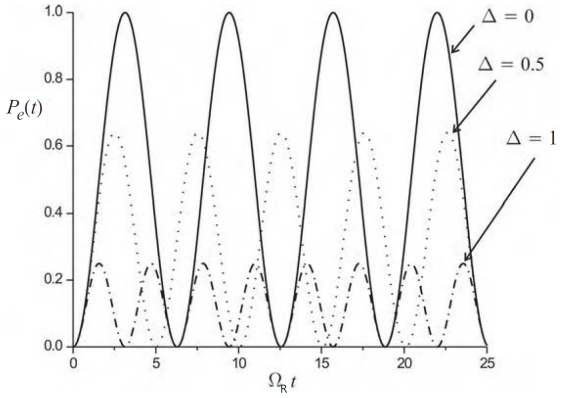

Figure 4.1: Population of the excited state as a function of time

4.2.2 Fermi's Golden Rule

If the external field is small, one can apply the perturbation method (for example, see Chapter 5 of Ref. [**Saku1994**]) to obtain (or from Eq. (4.2.12))

$$P_c(t) = |C_c|^2 = \frac{|V_0|^2 \sin^2 \frac{\Delta t}{2}}{\hbar^2 \Delta^2}. \tag{4.2.14}$$

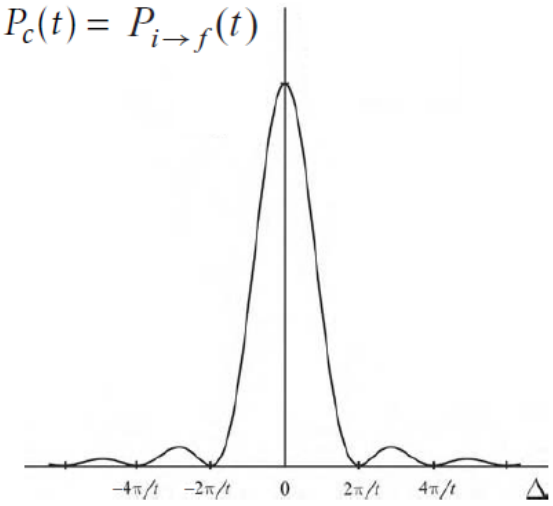

Figure 4.2: The transition probability $P_c(t)$ at a momentum t. When t is large, the function is approximately a delta function.

When t is large, the fraction is approximately a delta function

$$\frac{\sin^2 \frac{\Delta t}{2}}{\Delta^2} \simeq \frac{\pi t}{2} \delta(\Delta). \tag{4.2.15}$$

The transition rate $W_{v\to c}$ is

$$W_{v\to c} = \frac{P_c(t)}{t} = \frac{\pi}{2}\frac{|V_0|^2}{\hbar^2}\delta(\omega-\omega_{cv}) \tag{4.2.16}$$

$$= \frac{\pi}{2}\frac{|\mathbf{E}_0\cdot\mathbf{d}_{cv}|^2}{\hbar^2}\delta(\omega-\omega_{cv}) \tag{4.2.17}$$

$$= \frac{\pi}{2}\frac{|\langle c|\mathbf{H}_I|v\rangle|^2}{\hbar^2}\delta(\omega-\omega_{cv}), \tag{4.2.18}$$

which is the famous Fermi's Golden rule.

4.3 Classical Matter and Quantum Fields

Currents and charges are treated as classical numbers. Time-dependent charges and currents are not independent variables. They are related by the continuity equation. This assumption is adequate when currents come form a lot of electrons and the quantum fluctuations are ignored. The typical problem is how a current source $\mathbf{j}(\mathbf{r},t)$ interacts with photons. Thus, currents are given functions, and the problem is to solve filed Hamiltonian.

$$\mathcal{H} = \mathcal{H}_F + \mathcal{H}_I \tag{4.3.1}$$

$$= \sum_m \hbar\omega_m a_m^\dagger a_m - \sum_m \mathbf{E}_m\cdot\mathbf{d} \tag{4.3.2}$$

$$= \sum_m \hbar\omega_m a_m^\dagger a_m - \sum_m \left(\frac{\boldsymbol{\mathcal{E}}_m a + \boldsymbol{\mathcal{E}}_m^* a^\dagger}{2}\right)\cdot\mathbf{d}, \tag{4.3.3}$$

where we should treat the dipole as a classical function with a frequency ω ,

$$\mathbf{d} = \mathbf{d}_0 e^{-i\omega t} = \int dv\mathbf{P} = \int dv\mathbf{P}_0 e^{-i\omega t} \tag{4.3.4}$$

$$= -i\omega\int dv\mathbf{J}. \tag{4.3.5}$$

We can also use vector potential operators via the relation

$$\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{A}, \tag{4.3.6}$$

$$\mathbf{A} = \left(\frac{\boldsymbol{\mathcal{E}}_m a - \boldsymbol{\mathcal{E}}_m^* a^\dagger}{2i\omega}\right). \tag{4.3.7}$$

Considering a single mode and $\omega_m = \omega$, the Hamiltonian becomes

$$\mathcal{H} = \hbar\omega a^\dagger a - \int dv\mathbf{A}\cdot\mathbf{J}. \tag{4.3.8}$$

4.3.1 Generation of Coherent States

We are going to show a coherent state $|\alpha\rangle$ can be generate by a harmonic oscillating current density $\mathbf{J} = \mathbf{J}_0(\mathbf{r})e^{i\omega t}$. [3] The Hamiltonian becomes

$$\mathcal{H} = \hbar\omega a^\dagger a + \left(V_0 a + V_0^* a^\dagger\right), \tag{4.3.9}$$

[3] Physical currents should be real. We can think that $e^{i\omega t}$ comes from $\cos\omega t$

where

$$V_0 = i\omega e^{i\omega t} \int dv \boldsymbol{\mathcal{E}}_\omega(\mathbf{r}) \cdot \mathbf{J}_0(\mathbf{r}). \tag{4.3.10}$$

In the interaction picture, the interaction Hamiltonian becomes

$$\mathcal{H}_I = \left(V_I a + V_I^* a^\dagger\right), \tag{4.3.11}$$

where the interaction potenital is time-indepdenent and reads

$$V_I = i\omega \int dv \boldsymbol{\mathcal{E}}_\omega(\mathbf{r}) \cdot \mathbf{J}_0(\mathbf{r}). \tag{4.3.12}$$

The evolution of a state is given by

$$|\psi(t)\rangle_I = \hat{\mathcal{T}}[e^{-i\int \frac{\mathcal{H}_I(t)}{\hbar} dt}]|\psi(0)\rangle_I \tag{4.3.13}$$

where $\hat{\mathcal{T}}[]$ denotes the time-ordering[4]. In this case, the interaction Hamiltonian in the interaction picture is time-independent,

$$|\psi(t)\rangle_I = e^{-i\frac{\mathcal{H}_I(t)}{\hbar}t}|\psi(0)\rangle_I \tag{4.3.14}$$

$$= e^{\alpha^* a - \alpha a^\dagger}|\psi(0)\rangle_I, \tag{4.3.15}$$

where

$$\alpha = i\frac{V_I^*}{\hbar}t \tag{4.3.16}$$

$$= \frac{i\omega \int dv \boldsymbol{\mathcal{E}}_\omega^*(\mathbf{r}) \cdot \mathbf{J}_0^*(\mathbf{r})}{\hbar} t. \tag{4.3.17}$$

Equation (4.3.15) is indeed the displacement operator. If the initial state is the ground state $|0\rangle$, the final state is a coherent state,

$$|\psi(t)\rangle_I = e^{\alpha^* a - \alpha a^\dagger}|0\rangle \tag{4.3.18}$$

$$= |\alpha\rangle. \tag{4.3.19}$$

One interesting observation is that $|\alpha| \sim t$ and the photon number $n \sim t^2$ grows quadratically.

4.4 Fully Quantum Approach

Both matter and field are quantized. The Hamiltonian is

$$\mathcal{H} = \sum_m \hbar\omega_m a_m^\dagger a_m + \sum_n E_n |E_n\rangle\langle E_n| - \mathbf{E} \cdot \mathbf{d}, \tag{4.4.1}$$

[4]Time-ordering is necessary if H_I is time-dependent and $[H_I(t_1), H_I(t_2)] \neq 0$

where the electric field operator is

$$\mathbf{E} = \sum_m \frac{\boldsymbol{\mathcal{E}}_m a_m + \boldsymbol{\mathcal{E}}_m^* a_m^\dagger}{2}, \tag{4.4.2}$$

and the dipole matrix operator is

$$\begin{pmatrix} \mathbf{d}_{11} & \mathbf{d}_{12} & \dots \\ \mathbf{d}_{21} & \mathbf{d}_{22} & \\ \vdots & & \ddots \end{pmatrix}, \tag{4.4.3}$$

with $\mathbf{d}_{nn'} = \langle E_n|\mathbf{d}|E_{n'}\rangle$ and $\mathbf{d} = q\mathbf{r}$.

4.4.1 Two-Level System and Single-Mode Photons

The Hamiltonian is

$$\mathcal{H} = \hbar\omega a^\dagger a + \begin{pmatrix} E_c & 0 \\ 0 & E_v \end{pmatrix} - \mathbf{E}\cdot\mathbf{d}. \tag{4.4.4}$$

where the electric field operator is

$$\mathbf{E} = \frac{\boldsymbol{\mathcal{E}}_\omega a + \boldsymbol{\mathcal{E}}_\omega^* a^\dagger}{2}, \tag{4.4.5}$$

and the dipole matrix operator is

$$\begin{pmatrix} 0 & \mathbf{d}_{cv} \\ \mathbf{d}_{vc} & 0 \end{pmatrix}, \tag{4.4.6}$$

where we assume that the diagonal terms vanish. The transition rate from $|n\rangle|E_c\rangle$ to $|n+1\rangle|E_v\rangle$ is obtained by

$$W_{\text{emission}} = \frac{\pi}{2}\frac{|\langle n+1|\langle E_v|\mathbf{H}_I|n\rangle|E_c\rangle|^2}{\hbar^2}\delta(\omega-\omega_{cv}) \tag{4.4.7}$$

$$= \frac{(n+1)\pi}{2}\frac{|\boldsymbol{\mathcal{E}}_\omega\cdot\mathbf{d}_{cv}|^2}{\hbar^2}\delta(\omega-\omega_{cv}). \tag{4.4.8}$$

An interesting result occurs when $n = 0$. The emission is not zero when $n = 0$. This is the phenomenon "spontaneous emission". When $n > 0$, it corresponds to the stimulated emission. The transition rate from $|n\rangle|E_v\rangle$ to $|n-1\rangle|E_c\rangle$ is obtained by

$$W_{\text{absorption}} = \frac{\pi}{2}\frac{|\langle n-1|\langle E_c|\mathbf{H}_I|n\rangle|E_v\rangle|^2}{\hbar^2}\delta(\omega-\omega_{cv}) \tag{4.4.9}$$

$$= \frac{n\pi}{2}\frac{|\boldsymbol{\mathcal{E}}_\omega^*\cdot\mathbf{d}_{vc}|^2}{\hbar^2}\delta(\omega-\omega_{cv}). \tag{4.4.10}$$

4.4.2 Jaynes–Cummings Model

The TLS and single-mode photon Hamiltonian can be further simplified with the RWA,

The Hamiltonian is

$$\mathcal{H} = \hbar\omega a^\dagger a + \begin{pmatrix} E_c & 0 \\ 0 & E_v \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & \boldsymbol{\mathcal{E}}_\omega \cdot \mathbf{d}_{cv} a + \boldsymbol{\mathcal{E}}_\omega^* \cdot \mathbf{d}_{cv} a^\dagger \\ \boldsymbol{\mathcal{E}}_\omega \cdot \mathbf{d}_{vc} a + \boldsymbol{\mathcal{E}}_\omega^* \cdot \mathbf{d}_{vc} a^\dagger & 0 \end{pmatrix} \tag{4.4.11}$$

$$\simeq \hbar\omega a^\dagger a + \begin{pmatrix} E_c & 0 \\ 0 & E_v \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & \boldsymbol{\mathcal{E}}_\omega \cdot \mathbf{d}_{cv} a \\ \boldsymbol{\mathcal{E}}_\omega^* \cdot \mathbf{d}_{vc} a & 0 \end{pmatrix} \tag{4.4.12}$$

$$= \hbar\omega a^\dagger a + \frac{E_c + E_v}{2} + \frac{\hbar\omega_{cv}}{2}\sigma_z + \hbar\left(\lambda\sigma_+ a + \lambda^*\sigma_- a^\dagger\right) \tag{4.4.13}$$

where

$$\lambda = \frac{-1}{2}\boldsymbol{\mathcal{E}}_\omega \cdot \mathbf{d}_{cv}. \tag{4.4.14}$$

The average energy $\frac{E_c+E_v}{2}$ is only a constant so as irrelavent to dynamics. In most cases, it is possible to make λ real by choosing the phase of $\mathbf{d}_{cv}$. The Jaynes–Cummings Model is then obtained as

$$\mathcal{H}_{JC} = \hbar\omega a^\dagger a + \frac{\hbar\omega_{cv}}{2}\sigma_z + \hbar\lambda\left(\sigma_+ a + \sigma_- a^\dagger\right). \tag{4.4.15}$$

We have used the Pauli matrices

$$\sigma_z = |E_c\rangle\langle E_c| - |E_v\rangle\langle E_v| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.4.16}$$

$$\sigma_+ = |E_c\rangle\langle E_v| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{4.4.17}$$

$$\sigma_- = |E_v\rangle\langle E_c| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4.4.18}$$

The electron number operator is an identity,

$$N_e = |E_c\rangle\langle E_c| + |E_v\rangle\langle E_v|, \tag{4.4.19}$$

and the excitation number operator is

$$N_{ex} = |E_c\rangle\langle E_c| + a^\dagger a. \tag{4.4.20}$$

These numbers are conservative since the commutators vanish

$$[H, N_e] = 0, \tag{4.4.21}$$

$$[H, N_{ex}] = 0. \tag{4.4.22}$$

Exercise 4.1 Excitation Numberen Show Eq. (4.4.30). ■

4.4.3 Jaynes–Cummings Model

The Jaynes–Cummings Model is then obtained as

$$\mathcal{H}_{JC} = \hbar\omega a^\dagger a + \frac{\hbar\omega_{cv}}{2}\sigma_z + \hbar\lambda\left(\sigma_+ a + \sigma_- a^\dagger\right). \tag{4.4.23}$$

We have used the Pauli matrices

$$\sigma_z = |E_c\rangle\langle E_c| - |E_v\rangle\langle E_v| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.4.24}$$

$$\sigma_+ = |E_c\rangle\langle E_v| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{4.4.25}$$

$$\sigma_- = |E_v\rangle\langle E_c| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4.4.26}$$

The electron number operator is an identity,

$$N_e = |E_c\rangle\langle E_c| + |E_v\rangle\langle E_v|, \tag{4.4.27}$$

and the excitation number operator is

$$N_{ex} = |E_c\rangle\langle E_c| + a^\dagger a. \tag{4.4.28}$$

These numbers are conservative since the commutators vanish

$$[\mathcal{H}, N_e] = 0, \tag{4.4.29}$$

$$[\mathcal{H}, N_{ex}] = 0. \tag{4.4.30}$$

The Hamiltonian is decomposed as

$$\mathcal{H}_I = \hbar\omega N_{ex} - \hbar\frac{\omega}{2}N_e, \tag{4.4.31}$$

$$\mathcal{H}_{II} = -\frac{\hbar\Delta}{2}\sigma_z + \hbar\lambda\left(\sigma_+ a + \sigma_- a^\dagger\right). \tag{4.4.32}$$

with $\omega = \omega_{cv} + \Delta$. The two Hamiltonians $\mathcal{H}_I$ and $\mathcal{H}_{II}$ commute with each other,

$$[\mathcal{H}_I, \mathcal{H}_{II}] = 0, \tag{4.4.33}$$

which means the two Hamiltonian are decoupled and can be **block-diagonalized**. The Hamiltonian $\mathcal{H}_I$ describes the conservative numbers so that it is irrelevant to dynamics. All the dynamics is given by $\mathcal{H}_{II}$. We can use the interaction picture where $\mathcal{H}_0 = \mathcal{H}_I$ so that the dynamics is given by

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \mathcal{H}_{II}|\psi\rangle. \tag{4.4.34}$$

The kets here are in the interaction picture. The basis kets are

$$|E_m\rangle \otimes |n\rangle \equiv |E_m\rangle|n\rangle \tag{4.4.35}$$

where $n = c$ or v and $n = 0,\ 1,\ ,2,\ 3,...$ It seems that if we want to use the number states as the basis, the dimension of the Hamiltonian would be infinite. This is true, but the Hamiltonian can be block-diagonalized. Each block is just a 2 by 2 matrix. **Because the excitation number is conserved, only the states with the same excitation number are coupled.** For example, the state $|E_c\rangle|n\rangle$ is only coupled to $|E_v\rangle|n+1\rangle$. The problem is then to solve a two-dimensional Hamiltonian since each block is independent.

■ **Example 4.1** Number Statens Let the light in the number state $|n\rangle$. The two basis kets are

$$|E_v\rangle|n+1\rangle \equiv |i\rangle, \tag{4.4.36}$$
$$|E_c\rangle|n\rangle \equiv |f\rangle. \tag{4.4.37}$$

An arbitrary state in the interaction picture is

$$|\psi(t)\rangle = C_i(t)|i\rangle + C_f(t)|f\rangle. \tag{4.4.38}$$

Plugging this state in Eq. (4.4.34), we obtain

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} C_f \\ C_i \end{pmatrix} = \begin{pmatrix} -\frac{\hbar\Delta}{2} & \sqrt{n+1}\hbar\lambda \\ \sqrt{n+1}\hbar\lambda & \frac{\hbar\Delta}{2} \end{pmatrix}\begin{pmatrix} C_f \\ C_i \end{pmatrix}. \tag{4.4.39}$$

The eigenfrequencies are

$$\omega_\pm = \pm\sqrt{\frac{\Delta^2}{4} + (n+1)\lambda^2}. \tag{4.4.40}$$

and the eigenvectors (using the Bloch sphere representation) are

$$|\omega_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} e^{-i\omega_+ t} \tag{4.4.41}$$
$$|\omega_-\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix} e^{-i\omega_- t} \tag{4.4.42}$$

with

$$\theta = -\tan^{-1}\left(\frac{2\sqrt{n+1}\lambda}{\Delta}\right). \tag{4.4.43}$$

If the initial state is $C_i = 1$ and $C_f = 0$, the solution becomes

$$|\psi\rangle = \sin\frac{\theta}{2}|\omega_+\rangle - \cos\frac{\theta}{2}|\omega_-\rangle, \tag{4.4.44}$$
$$C_i(t) = \cos\omega_+ t + i\cos\theta\sin\omega_+ t, \tag{4.4.45}$$
$$C_f(t) = -i\sin\theta\sin\omega_+ t. \tag{4.4.46}$$

The population of the excited state $n_e = |C_f(t)|^2$ is

$$n_e = \sin^2\theta\sin^2\omega_+ t, \tag{4.4.47}$$
$$= \sin^2\theta\sin^2\sqrt{\frac{\Delta^2}{4} + (n+1)\lambda^2}t. \tag{4.4.48}$$

This is the Rabi oscillation between the states $|E_v\rangle|n+1\rangle$ and $|E_c\rangle|n\rangle$. Only when the detuning is zeros, we have $\sin\theta = 1$ and the maximum excitation. The Rabi frequency is

$$\omega_+ = \sqrt{\frac{\Delta^2}{4} + (n+1)\lambda^2}. \tag{4.4.49}$$

The Rabi frequency does depend on the number of the photons. One novel case is $n = 0$ where the frequency is not zero but

$$\omega_+(n=0) = \sqrt{\frac{\Delta^2}{4} + \lambda^2}. \tag{4.4.50}$$

This means that there exists the Rabi oscillation even when there is no photon.[5] This is called the "vacuum Rabi oscillations".

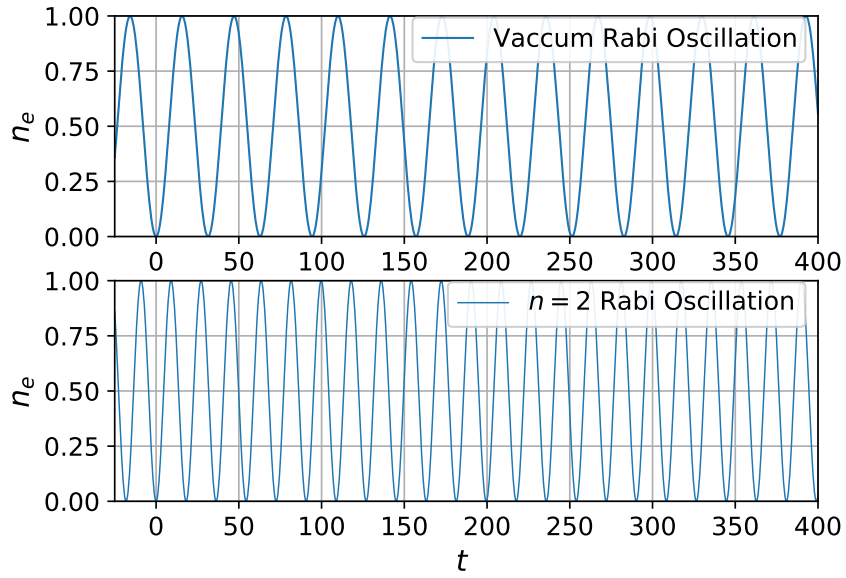

Figure 4.3: Rabi oscillations of the JC models for $n = 0$ and $n = 2$. The other parameters are $\Delta = 0$ and $\lambda = 0.1$

.

■

4.4.4 JC models with a Coherent State

Let us consider a more general situation where the photon state is

$$|\text{field}\rangle = \sum_{n=0}^{\infty} C_n |n\rangle, \tag{4.4.51}$$

and the two level system is

$$|\text{TSL}\rangle = C_c|E_c\rangle + C_v|E_v\rangle. \tag{4.4.52}$$

The total state is

$$|\psi\rangle = |\text{TSL}\rangle \otimes |\text{field}\rangle. \tag{4.4.53}$$

[5] Well, the average number of photons is 1/2.

The solution is then (when $\Delta = 0$)

$$|\psi\rangle = \sum_n [C_c C_n \cos(\omega_{n+1} t) - i C_v C_{n+1} \sin(\omega_{n+1} t)] |E_c\rangle |n\rangle \tag{4.4.54}$$

$$+ \sum_n [C_v C_{n+1} \cos(\omega_{n+1} t) - i C_c C_n \sin(\omega_{n+1} t)] |E_v\rangle |n+1\rangle, \tag{4.4.55}$$

where

$$\omega_n = \omega_+(n). \tag{4.4.56}$$

Let the initial state be $C_c = 0$ and $C_v = 1$. The population of the excited state is

$$n_e = |C_c(t)|^2 = \sum_n |C_{n+1}|^2 \sin^2 \omega_{n+1} t \tag{4.4.57}$$

$$= \sum_n |C_{n+1}|^2 \left(\frac{1 - \cos 2\omega_{n+1} t}{2} \right) \tag{4.4.58}$$

$$= \frac{1}{2} - \sum_n |C_{n+1}|^2 \left(\frac{\cos 2\omega_{n+1} t}{2} \right). \tag{4.4.59}$$

In terms of n, we obtain

$$n_e = \frac{1}{2} - \sum_n |C_{n+1}|^2 \left(\frac{\cos 2\lambda \sqrt{n+1}\, t}{2} \right). \tag{4.4.60}$$

Figure 4.4 shows the populations in the cases of coherent states. Even with a coherent state, the population is not a simple harmonic oscillation as in the classical case. There are two new properties. First, the oscillation lasts for a time τ_c (the duration of the wave packet.) and **collapses**. It is shown that the time τ_c is in the limit $n \to \infty$,

$$\tau_c \simeq \frac{\sqrt{2}}{\lambda}. \tag{4.4.61}$$

After a rephasing time τ_{rp}, the oscillation comes back. This is called the **revival**. The time τ_{rp} is in the limit $n \to \infty$,

$$\tau_{\mathrm{rp}} \simeq \frac{4\pi |\alpha|}{\lambda}. \tag{4.4.62}$$

Two properties of the JC model are

- Collapsing
- Revival

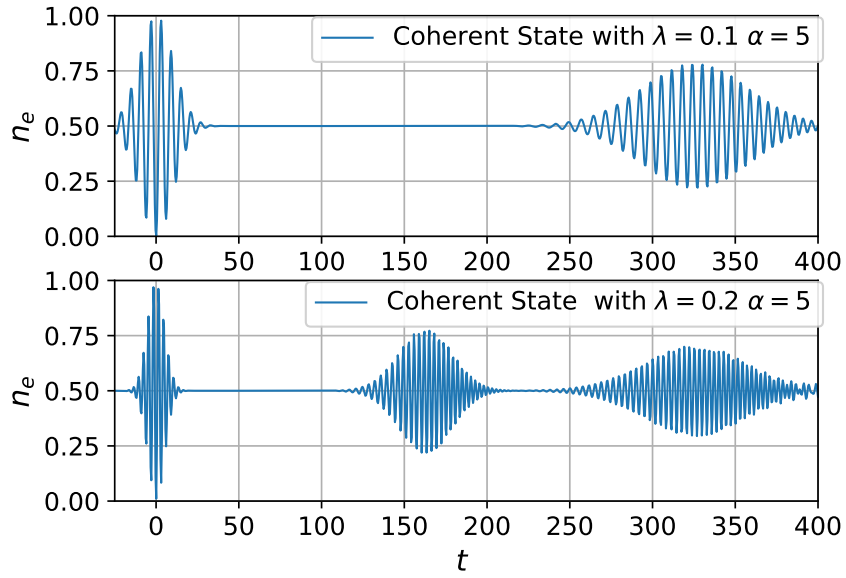

Figure 4.4: Rabi oscillations of the JC models for a coherent state. Collapsing and revival appear.

4.4.5 Dressed States

We focused on the dynamics of the JC model. Now, we discuss the eigenstates of the JC model. First, the photon energy in the vaccuum is $E = n\hbar\omega$.[6] In a cavity, photons are coupled with the TLS. As a result, the photon energies are shifted. We can think that the combination of photons and the TLS leads to a new state called "dressed states", or in the context of condensed matter physics, "polaritons". We start with the full Hamiltonian,

$$\mathcal{H} = \hbar\omega a^\dagger a - \hbar\Delta\sigma_z + \hbar\lambda(\sigma_- a^\dagger + \sigma_+ a). \tag{4.4.63}$$

Consider the subspace spanned by Eqs. (4.4.36) and (4.4.37). The eigenvalues are

$$E_{1n} = n\hbar\omega + \omega_n, \tag{4.4.64}$$
$$E_{2n} = n\hbar\omega - \omega_n, \tag{4.4.65}$$

where $\omega_n = \sqrt{\frac{\Delta^2}{4} + (n+1)\lambda^2}$ and the eigenvectors (using the Bloch sphere representation) are

$$|1n\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} e^{-i\omega_+ t} \tag{4.4.66}$$
$$|2n\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix} e^{-i\omega_- t} \tag{4.4.67}$$

with

$$\theta = -\tan^{-1}\left(\frac{2\sqrt{n+1}\lambda}{\Delta}\right). \tag{4.4.68}$$

[6] We drop $1/2\hbar\omega$.

The dressed photons are the eigenstates of the total system. Compared to photons in vacuum, their frequencies shift and become non-degenerate. The splitting of dressed states is the origin of the Mollow triplet emissions.

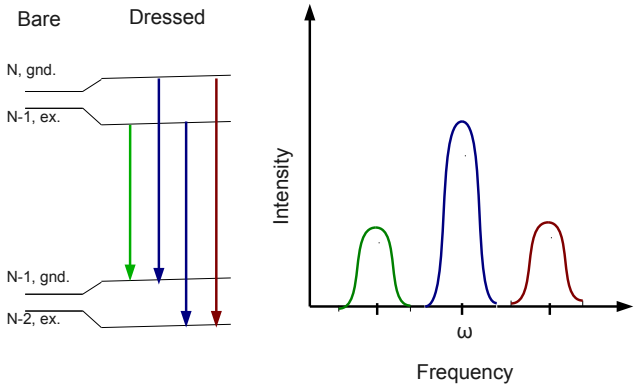

Figure 4.5: Mollow triplet emissions.

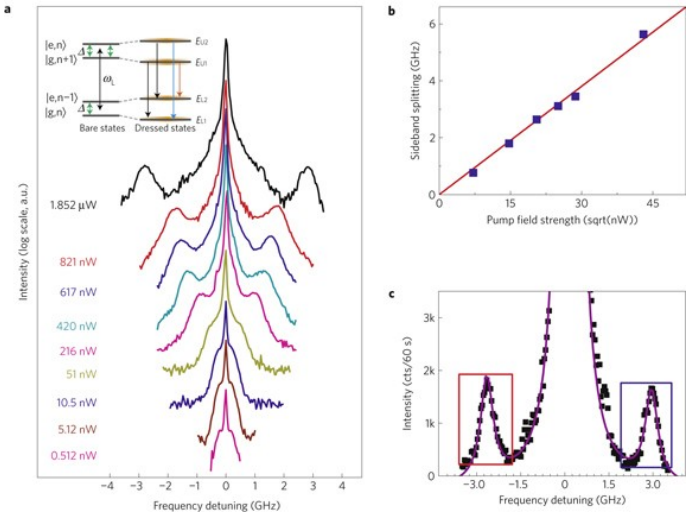

Figure 4.6: Experimental observation of the Mollow triplet emissions. From Nature Physics 5, 198–202(2009)

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[1] J. J. Sakurai, *Modern Quantum Mechanics*, 1994 and 2010

5. Correlation and Coherence Functions

5.1 Correlation Functions

A correlation function is a mathematical tool used to measure the dependence between two or more variables. In physics, these variables are physical quantities depending on positions and time. For example, let $s_1(\mathbf{r},t)$ and and $s_2(\mathbf{r},t)$ be the two amplitudes of two scalar waves. The first order correlation function is

$$C^{(1)}(\Delta\mathbf{r},\Delta t) = \langle s_1^*(\mathbf{r}_1,t_1)s_2(\mathbf{r}_2,t_2)\rangle, \tag{5.1.1}$$

where $\langle ...\rangle$ denotes an ensemble average.

The correlation functions are used to describe spatial and temporal coherences of waves. The superposition of two waves is

$$|s_1(\mathbf{r}_1,t_1)+s_2(\mathbf{r}_2,t_2)|^2 = |s_1(\mathbf{r}_1,t_1)|^2+|s_2(\mathbf{r}_2,t_2)|+2\mathrm{Re}[s_1(\mathbf{r}_1,t_1)s_2^*(\mathbf{r}_2,t_2)]. \tag{5.1.2}$$

The ensemble-averaged interference is

$$\langle 2\mathrm{Re}[s_1(\mathbf{r}_1,t_1)s_2^*(\mathbf{r}_2,t_2)]\rangle = 2\mathrm{Re}[C(\Delta\mathbf{r},\Delta t)]. \tag{5.1.3}$$

Correlation functions are called auto-correlation functions if s_1 and s_2 are the same variables. If s_1 is the same source as s_2 and $\Delta\mathbf{r}=0$, the correlation functions measure the temporal coherence. If s_1 is the same source as s_2 and $\Delta t=0$, the correlation functions measure the spatial coherence. We can define the dimensionless correlations function $g^{(1)}$, [1] called the first-order correlation function or normalized correlation function,

$$g^{(1)}(\Delta\mathbf{r},\Delta t) = \frac{\langle s_1^*(\mathbf{r}_1,t_1)s_2(\mathbf{r}_2,t_2)\rangle}{\sqrt{\langle|s_1(\mathbf{r}_1,t_1)|^2\rangle\langle|s_2(\mathbf{r}_2,t_2)|^2\rangle}}. \tag{5.1.4}$$

[1] In the literature, people use $\gamma^{(1)}$ for classical cases and $g^{(1)}$ for quantum cases. Here, I use $g^{(1)}$ for both the cases.

According to the Schwartz inequality, $0 \leq g^{(1)} \leq 1$. The coherences are related $g^{(1)}$ by

$$|g^{(1)}| = 1, \qquad \text{coherent}, \tag{5.1.5}$$

$$0 < |g^{(1)}| < 1, \qquad \text{partially coherent}, \tag{5.1.6}$$

$$|g^{(1)}| = 0, \qquad \text{completely incoherent}. \tag{5.1.7}$$

The coherence function $g^{(1)}$ typically decreases at time goes or traveled optical length increases. The process is called decoherence. Two main sources of decoherence are (a) non-monochromatic light and (b) noises due to collisions or scatterings. We might model the decoherence as

$$g^{(1)}(t) = g^{(1)}(0)\exp\left(-\frac{t}{\tau_c}\right), \tag{5.1.8}$$

where τ_c is the coherence time. If a light source is not monochromatic and has a band width $\Delta\omega$, the coherent time is $\tau_c \sim \frac{1}{\Delta\omega}$.

In probability and statistics theories, we can specify a probability distribution of a variable X if we know all the moments of X, i.e., $\langle X\rangle$, $\langle X^2\rangle$, $\langle X^3\rangle$,... If we want to fully specify the relation between X and Y, we need to know not only $\langle XY\rangle$ but also the higher order terms such as $\langle X^2Y^2\rangle$, $\langle X^3Y^3\rangle$. One can define the high-order autocorrelation functions are defined as

$$C^{(2)}(x_1, x_2, x_3, x_4) = \langle s^*(x_1)s^*(x_2)s(x_3)s(x_4)\rangle \tag{5.1.9}$$

ans so on. Here, x_n denotes (x_1, t_1). One useful case is

$$C^{(2)}(x_1, x_2, x_2, x_1) = \langle I(x_1)I(x_2)\rangle, \tag{5.1.10}$$

which is called the intensity-intensity correlation function. The second-order coherence function $g^{(2)}$ is defined as

$$g^{(2)}(x_2 - x_1) \equiv g^{(2)}(x_1, x_2, x_2, x_1) = \frac{C^{(2)}(x_1, x_2, x_2, x_1)}{C^{(1)}(x_1, x_1)C^{(1)}(x_2, x_2)}. \tag{5.1.11}$$

In quantum optics, waves are electric fields. The scalar field $s(x)$ is replaced by $\mathcal{E} \equiv \boldsymbol{\mathcal{E}} \cdot \hat{e}$, where $\hat{e}$ is a unit vector. For example, the first-order coherence function becomes

$$g^{(1)}(\Delta\mathbf{r}, \Delta t) = \frac{\langle \mathcal{E}_1^*(\mathbf{r}_1, t_1)\mathcal{E}_2(\mathbf{r}_2, t_2)\rangle}{\sqrt{\langle|\mathcal{E}_1(\mathbf{r}_1, t_1)|^2\rangle\langle|\mathcal{E}_2(\mathbf{r}_2, t_2)|^2\rangle}}. \tag{5.1.12}$$

5.1.1 Definitions in Quantum Optics

When defining correlation functions for quantum optics, we have the following term

$$\langle \mathcal{E}_1^*(\mathbf{r}_1, t_1)\mathcal{E}_2(\mathbf{r}_2, t_2)\rangle. \tag{5.1.13}$$

Classically, the order of the product in the average does not matter. But quantumly, we have to deal with the order carefully. Physically, correlations are measured quantities. Measurement processes consist of absorptions of photons by the detectors. Say, first the

detector absorbed one photon at t_1 and another photon at a latter time t_2. This process is described by two annihilation operators

$$a(t_2)a(t_1)|i\rangle, \tag{5.1.14}$$

where $|i\rangle$ is the initial state. The probability of the process is proportional to the norm of Eq. (5.1.14).

$$\langle i|a^\dagger(t_1)a^\dagger(t_2)a(t_2)a(t_1)|i\rangle. \tag{5.1.15}$$

Thus, we have the following summary

- All annihilation operators are on the right.
- All creation operators are on the left.
- An annihilation operator at an earlier time is on the right.
- An creation operator at an earlier time is on the left.

The correlation functions are given by the density-matrix approach

$$\begin{aligned} C^{(1)} &= \mathrm{Tr}[\rho\mathcal{E}^*(t_2)a^\dagger(t_2)\mathcal{E}(t_1)a(t_1)] && (5.1.16)\\ &= \mathcal{E}^*(t_2)\mathcal{E}(t_1)\mathrm{Tr}[\rho a^\dagger(t_2)a(t_1)], && (5.1.17)\\ C^{(2)} &= \mathrm{Tr}[\rho\mathcal{E}^*(t_1)a^\dagger(t_1)\mathcal{E}^*(t_2)a^\dagger(t_2)\mathcal{E}(t_2)a(t_2)\mathcal{E}(t_1)a(t_1)] && (5.1.18)\\ &= \mathcal{E}^*(t_1)\mathcal{E}^*(t_2)\mathcal{E}(t_2)\mathcal{E}(t_1)\mathrm{Tr}[\rho a^\dagger(t_1)a^\dagger(t_2)a(t_2)a(t_1)] && (5.1.19)\\ &= \tilde{I}(t_1)\tilde{I}(t_2)\mathrm{Tr}[\rho a^\dagger(t_1)a^\dagger(t_2)a(t_2)a(t_1)]. && (5.1.20) \end{aligned}$$

We should use these correlation function to calculate the quantum coherence functions defined in Eqs. (5.1.11) and (5.1.12).

5.2 Young's Interference and First-Order Coherent Function

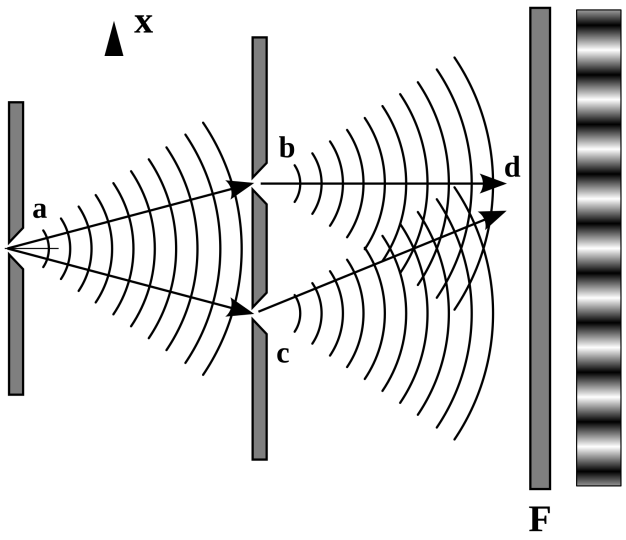

Figure 5.1: Young's interference.

Let the source generated single-mode photons whose annihilation operator is a. The photons then pass the two slits. The two slits are regarded as the light sources whose annihilation operators are a_1 and a_2. We assume the two slits are equal such that

$$a = \frac{1}{\sqrt{2}}(a_1 + a_2). \tag{5.2.1}$$

The intensity on the screen is indeed the first-order correlation function

$$I(t) = \tilde{I}(t)\mathrm{Tr}[\rho a^\dagger(t)a(t)] \tag{5.2.2}$$

$$= \frac{\tilde{I}(t)}{2}\left\{\mathrm{Tr}[\rho a_1^\dagger(t)a_1(t)] + \mathrm{Tr}[\rho a_2^\dagger(t)a_2(t)] + \mathrm{Tr}[\rho a_1^\dagger(t)a_2(t)] + \mathrm{Tr}[\rho a_2^\dagger(t)a_1(t)]\right\} \tag{5.2.3}$$

$$= \frac{\tilde{I}(t)}{2}\left\{\mathrm{Tr}[\rho a_1^\dagger(0)a_1(0)] + \mathrm{Tr}[\rho a_2^\dagger(0)a_2(0)] + \mathrm{Tr}[\rho a_1^\dagger(0)a_2(0)]e^{i\Phi} + \mathrm{Tr}[\rho a_2^\dagger(0)a_1(0)]e^{-i\Phi}\right\}, \tag{5.2.4}$$

where Φ is the phase difference due to the optical length. When the incident light is a one-photon state, we have the state after two slits

$$a^\dagger|0\rangle = \frac{1}{\sqrt{2}}\left(a_1^\dagger + a_2^\dagger\right)|0\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle). \tag{5.2.5}$$

The first-order correlation function is

$$I(t) = \tilde{I}(t)\left(\frac{1+\cos\Phi}{2}\right). \tag{5.2.6}$$

When the incident light is a two-photon state, we have the state after two slits

$$(a^\dagger)^2|0\rangle = \frac{1}{2}\left(a_1^\dagger + a_2^\dagger\right)^2|0\rangle = \frac{1}{2}(|20\rangle + \sqrt{2}|11\rangle + |02\rangle). \tag{5.2.7}$$

The first-order correlation function is

$$I(t) = 2\tilde{I}(t)\left(\frac{1+\cos\Phi}{2}\right). \tag{5.2.8}$$

For a n-photon state, we have

$$I(t) = n\tilde{I}(t)\left(\frac{1+\cos\Phi}{2}\right). \tag{5.2.9}$$

It can be shown that for a coherent state, we have

$$I(t) = \bar{n}\tilde{I}(t)\left(\frac{1+\cos\Phi}{2}\right). \tag{5.2.10}$$

We have the following notes

- Interference occurs even when for a single-photon state.
- First-order correlation functions describe interferences.
- First-order correlation functions **can not** distinguish what kind of photon state the light is. For number states and coherent states, the interferences are the same as $\cos\Phi$.

5.3 Hanbury-Brown and Twiss Experiment and g_2

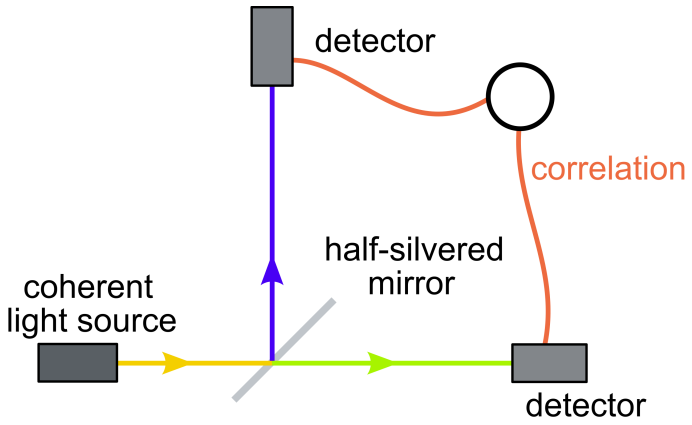

Figure 5.2: Hanbury- Brown and Twiss experiment.

First-order coherence in Young's experiment can determine how monochromatic a light is, or noise levels by measuring coherent lengths. The properties revealed by first-order coherence are related to the modes and the environments (noise). However, first-order coherence does not tell which photon state it is, that is, if two states are from the same mode, first-order coherence can not distinguish their photon distribution. Say, to distinguish a number state $|n\rangle$ and a coherent state $|\alpha\rangle$ from the same mode, we have to use the second-order coherence function $g^{(2)}$.

In the 1950s, Hanbury Brown and Twiss developed an experiment to measure intensity-intensity correlations (see Fig. 5.2). The main difference from a Young's interference is that there are two intensity detectors instead of one. The two optical paths after the beam splitter lead to a time delay τ. The rates of coincident events is measured by the coincidence counter and proportional to

$$C^{(2)} = \langle \mathcal{E}^*(t_1)a^\dagger(t_1)\mathcal{E}^*(t_2)a^\dagger(t_2).\mathcal{E}(t_2)a(t_2)\mathcal{E}(t_1)a(t_1)\rangle \tag{5.3.1}$$

5.3.1 Classical Regime

Classically, we can write the second-order correlation function as

$$C^{(2)} = \langle I(t+\tau)I(t)\rangle, \tag{5.3.2}$$

and the second-order coherent function $g^{(2)}$ as

$$g^{(2)} = \frac{\langle I(t+\tau)I(t)\rangle}{\langle I(t+\tau)\rangle\langle I(t)\rangle}. \tag{5.3.3}$$

For a stationary light, $\langle I(t)\rangle$ is time-independent. The second-order coherent function $g^{(2)}$ becomes

$$g^{(2)} = \frac{\langle I(t+\tau)I(t)\rangle}{\langle I(t)\rangle^2}. \tag{5.3.4}$$

The following properties can be shown (only in the classical regime),

$$1 \leq g^{(2)}(0) < \infty, \tag{5.3.5}$$
$$g^{(2)}(\tau) \leq g^{(2)}(0). \tag{5.3.6}$$

For a chaotic light source, it can be shown that

$$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2, \tag{5.3.7}$$

and if $g^{(1)}(\tau) = e^{-\frac{\tau}{\tau_c}}$,

$$g^{(2)}(\tau) = 1 + e^{-\frac{2\tau}{\tau_c}}. \tag{5.3.8}$$

For chaotic lights, it is shown $g^{(2)}(0) = 2$. This is called the photon bunching-effects.

5.3.2 Quantum Regime

The orders of operators in the correlation functions should be treated carefully. For a single mode, the second-order coherent function is

$$g^{(2)}(\tau) = \frac{\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\rangle}{\langle a^\dagger(t+\tau)a(t+\tau)\rangle\langle a^\dagger(t)a(t)\rangle} \tag{5.3.9}$$
$$= \frac{\langle a^\dagger a^\dagger aa\rangle}{\langle a^\dagger a\rangle^2} \tag{5.3.10}$$
$$= 1 + \frac{\sigma^2(n) - \bar{n}}{\bar{n}^2} \tag{5.3.11}$$

We can show that

$$g^{(2)}(0) = \begin{cases} 2, & \text{chaotic,} \\ 1, & \text{coherent,} \end{cases} \tag{5.3.12}$$

and for a number state

$$g^{(2)}(0) = \begin{cases} 0, & n = 0,\ 1, \\ \frac{n-1}{n}, & \text{else.} \end{cases} \tag{5.3.13}$$

As time increases, light becomes incoherent. We may use the result for chaotic light,

$$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2, \tag{5.3.14}$$

and $g^{(1)}(\tau) \to 0$ as $\tau \to \infty$. Thus as $\tau \to \infty$, $g^{(2)}(\tau) = 1$.

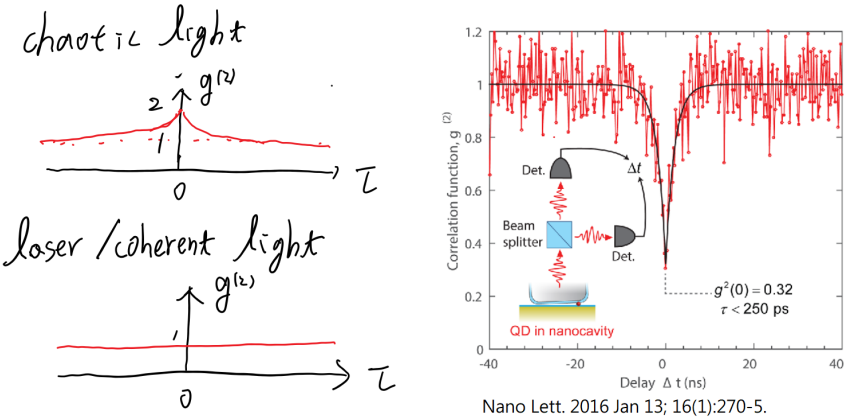

Figure 5.3: $g^{(2)}(\tau)$ of various photon ensembles.

Exercise 5.1 Coherence Fucntionscf Calculate $g^{(2)}(0)$ for the following cases:

a state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle). \tag{5.3.15}$$

(b) an ensemble

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|. \tag{5.3.16}$$

(c) an ensemble

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| + \frac{i}{2}|0\rangle\langle 1| + \frac{-i}{2}|1\rangle\langle 0| \tag{5.3.17}$$

Hint: use

$$g^{(2)}(0) = \frac{\langle a^\dagger a^\dagger a a\rangle}{\langle a^\dagger a\rangle^2} \tag{5.3.18}$$

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6. Beam Spliter and Non-Classical Light

6.1 Beam Splitters

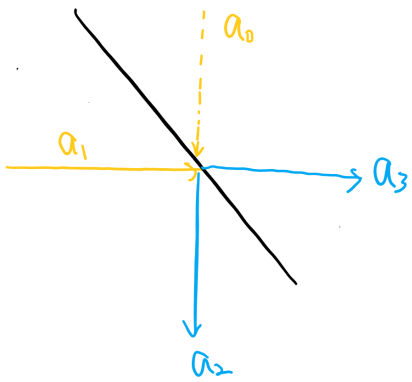

Figure 6.1: Beam spliter. Quantum descriptions.

Let the right-going photons have the annihilation operator a_1, and the left-going photons have the annihilation operator a_0. A beam splitter is described by the scattering matrix

$$\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}, \tag{6.1.1}$$

where $|r|^2 + |t|^2 = 1$. One can check that if

$$[a_0, a_0^\dagger] = 1, \tag{6.1.2}$$

$$[a_1, a_1^\dagger] = 1, \tag{6.1.3}$$

$$[a_0, a_1^\dagger] = 0, \tag{6.1.4}$$

we have

$$[a_i, a_j^\dagger] = \delta_{ij}. \tag{6.1.5}$$

The matrix is indeed a scattering matrix

$$U = \begin{pmatrix} t & r \\ r & t \end{pmatrix}, \tag{6.1.6}$$

which is unitary.

6.1.1 Single Photon

The incident state is $|0\rangle_0|1\rangle_1$, and

$$|0\rangle_0|1\rangle_1 = a_1^\dagger|0\rangle_0|0\rangle_1. \tag{6.1.7}$$

Aftering incident on the beam splitter, the state becomes

$$Ua_1^\dagger|0\rangle_0|0\rangle_1 = (ra_2 + ta_3)|0\rangle_2|0\rangle_3 \tag{6.1.8}$$

$$= r|1\rangle_2|0\rangle_3 + t|0\rangle_2|1\rangle_3, \tag{6.1.9}$$

which is an entangled state.

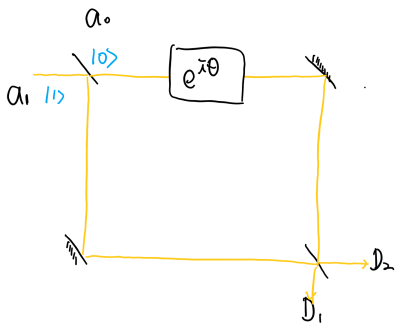

Figure 6.2: Mach–Zehnder interferometer.

Consider a Mach–Zehnder interferometer with two 50:50 beam splitters of $r = \frac{i}{\sqrt{2}}$ and $t = \frac{1}{\sqrt{2}}$. Let the initial state be $|0\rangle_0|1\rangle_1$. After the first beam splitter, the state becomes

$$\frac{i}{\sqrt{2}}|1\rangle_2|0\rangle_3 + \frac{1}{\sqrt{2}}|0\rangle_2|1\rangle_3. \tag{6.1.10}$$

When the state arrives at the second beam splitter, the state becomes

$$\frac{i}{\sqrt{2}}|1\rangle_2|0\rangle_3 + \frac{e^{i\theta}}{\sqrt{2}}|0\rangle_2|1\rangle_3, \tag{6.1.11}$$

where θ is a phase shift due to the difference of the two paths. The final state after the second beam splitter is (see the figure: a_2 is the new a_1 and a_3 is the new a_0)

$$\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{e^{i\theta}}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} \tag{6.1.12}$$

$$= \begin{pmatrix} \frac{(e^{i\theta}-1)}{2} \\ \frac{i(e^{i\theta}+1)}{2} \end{pmatrix} \tag{6.1.13}$$

The probability at D_1 is $\left|\frac{(e^{i\theta}-1)}{2}\right|^2 = \sin^2\frac{\theta}{2}$. The probability at D_2 is $\left|\frac{i(e^{i\theta}+1)}{2}\right|^2 = \cos^2\frac{\theta}{2}$. This is a more rigorous description of a single-photon interference.

6.1.2 N-Photons

Let the initial state be $|0\rangle_0|N\rangle_1 = \frac{(a_1^\dagger)^N}{\sqrt{N!}}|0\rangle$. After a beam splitter, the state becomes

$$\frac{(Ua_1^\dagger)^N}{\sqrt{N!}}|0\rangle = \frac{(ta_2^\dagger + ra_3^\dagger)^N}{\sqrt{N!}}|0\rangle. \tag{6.1.14}$$

6.1.3 Coherent States

Let the initial state be $|0\rangle_0|\alpha\rangle_1 = D_1[\alpha]|0\rangle = e^{\alpha a_1^\dagger - \alpha^* a_1}|0\rangle$. After a beam splitter, the state becomes

$$\begin{aligned} e^{\alpha U a_1^\dagger - \alpha^* U^\dagger a_1}|0\rangle &= e^{\alpha t a_2^\dagger - \alpha^* t^* a_2} e^{\alpha r a_3^\dagger - \alpha^* r^* a_3}|0\rangle && (6.1.15)\\ &= D_2[t\alpha]D_3[r\alpha]|0\rangle && (6.1.16)\\ &= |t\alpha\rangle_2|r\alpha\rangle_3. && (6.1.17)\end{aligned}$$

The input of a coherent state is split into a product of two coherent states. Unlike the single-photon case, this state is not entangled.

Consider a Mach–Zehnder interferometer with two 50:50 beam splitters of $r = \frac{i}{\sqrt{2}}$ and $t = \frac{1}{\sqrt{2}}$. Let the initial state be $|0\rangle_0|\alpha\rangle_1$. After the first beam splitter, the state becomes

$$\left|\frac{i\alpha}{\sqrt{2}}\right\rangle_2 \left|\frac{\alpha}{\sqrt{2}}\right\rangle_3. \tag{6.1.18}$$

When the state arrives at the second beam splitter, the state becomes

$$\left|\frac{i\alpha}{\sqrt{2}}\right\rangle_2 \left|\frac{e^{i\theta}\alpha}{\sqrt{2}}\right\rangle_3. \tag{6.1.19}$$

where θ is a phase shift due to the difference of the two paths. The final state after the second beam splitter is (see the figure: a_2 is the new a_1 and a_3 is the new a_0)

$$\left|\frac{\left(e^{i\theta}-1\right)}{2}\right\rangle_2 \left|\frac{i\left(e^{i\theta}+1\right)}{2}\right\rangle_3. \tag{6.1.20}$$

The intensity at D_1 is $|\alpha|^2\left|\frac{e^{i\theta}-1}{2}\right|^2 = \sin^2\frac{\theta}{2}|\alpha|^2$. The intensity at D_2 is $|\alpha|^2\left|\frac{e^{i\theta}+1}{2}\right|^2 = \cos^2\frac{\theta}{2}|\alpha|^2$. The two output beams are both coherent states. Thus, the phase θ can be obtained by

$$\frac{I_2 - I_1}{|\alpha|^2} = \cos\theta. \tag{6.1.21}$$

However, the amplitudes have uncertainty $\sigma(n) = \sqrt{\bar{n}}$. Thus, the phase has the uncertainty $\sigma(\theta) \sim \frac{1}{\sqrt{\bar{n}}}$. In experiments, we would like to use a coherent light (laser) with a well defined phase and a strong intensity such that the uncertainty in phase is small. But a strong-intensity light may lead to more noise such as radiation pressures, thermo noise, and so on. To solve this deli-ma, lights with small $\sigma(n)$ is used. These lights are non-classical lights.

6.2 Quadrature Squeezing

The quadrature operators X and Y, satisfy

$$[X, Y] = \frac{i}{2} \tag{6.2.1}$$

$$\Rightarrow \sigma(X)\sigma(Y) \geq \frac{1}{4}. \tag{6.2.2}$$

The coherent states satisfy the minimum uncertainty equations,

$$\sigma(X)\sigma(Y) = \frac{1}{4} \tag{6.2.3}$$

and

$$\sigma(X) = \sigma(Y) = \frac{1}{2}. \tag{6.2.4}$$

which is a circle in the phase space. The conditions of a quadrature squeezing are

$$\sigma(X) < \frac{1}{2} \text{ or } \sigma(Y) < \frac{1}{2} \tag{6.2.5}$$

while keeping $\sigma(X)\sigma(Y) = \frac{1}{4}$. Pictorially, a squeezed state is an ellipse in the phase space with a area $\frac{\pi}{16}$. Of course, we can squeeze a state in any direction other than X or Y. We can define the rotated quadrature operator as the following

$$X'(\theta) = \frac{ae^{-i\theta} + a^\dagger e^{i\theta}}{2}, \tag{6.2.6}$$

$$Y'(\theta) = \frac{ae^{-i\theta} - a^\dagger e^{i\theta}}{2i}, \tag{6.2.7}$$

which represent a coordinate transform of the quadrature operators. Depending on the squeezed axis, we have the following squeezed states. Question: which one has the minimum uncertainty of the photon number?

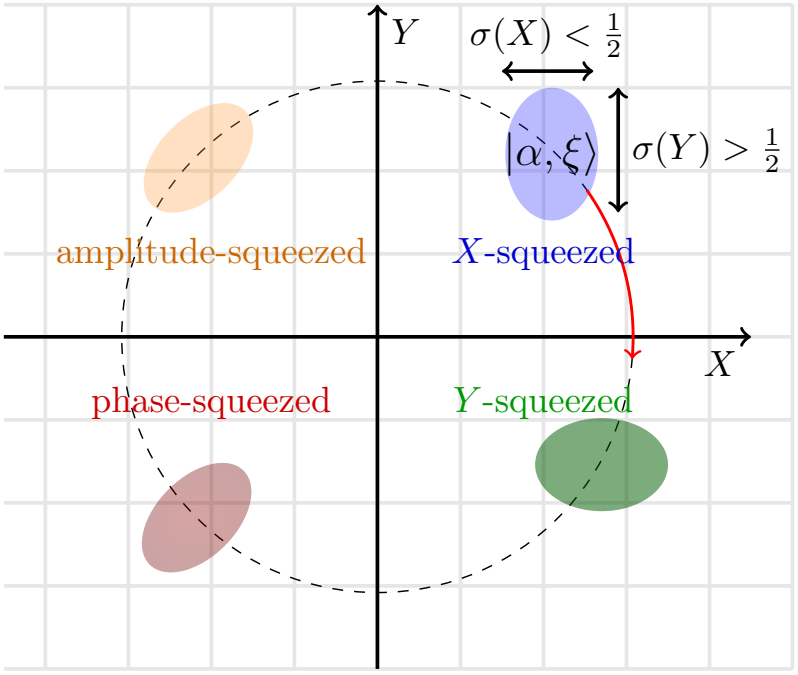

Figure 6.3: Squeezed States.

6.2.1 Squeezed Operators

Mathematically, a coherent state is generated by shifting a vacuum state in the phase space. This is done by the displacement operator $D(\alpha)$,

$$|\alpha\rangle = D(\alpha)|0\rangle. \tag{6.2.8}$$

We have shown that $D(\alpha)$ is the evolution operator U of a oscillating current source, that is, such a source creates a coherent state.

A squeezed state is generated by a squeeze operator,

$$S(\xi) = \exp\left(\frac{\xi^* a^2 - \xi (a^\dagger)^2}{2}\right), \tag{6.2.9}$$

where $\xi = re^{i\theta}$, and r is the squeeze parameter. A squeezed operator is a unitary operator. In principle, a unitary operator correspond a physical process. Observing the quadratic terms of the creation and annihilation operators, it is straightforward to speculate that **the physical processes are nonlinear**. This is because the quadratic terms come from the square of the electric field operators, $\mathbf{E}^2 = \left(\frac{\boldsymbol{\varepsilon} a + \boldsymbol{\varepsilon}^* a^\dagger}{2}\right)^2$. Squeeze operators have the relations

$$S^\dagger(\xi) S(\xi) = \mathbb{1}, \tag{6.2.10}$$

$$S^\dagger(\xi) a S(\xi) = a \cosh r - a^\dagger e^{i\theta} \sinh r, \tag{6.2.11}$$

$$S^\dagger(\xi) a^2 S(\xi) = \left(a \cosh r - a^\dagger e^{i\theta} \sinh r\right)^2, \tag{6.2.12}$$

$$S^\dagger(\xi) a^\dagger S(\xi) = a^\dagger \cosh r - a e^{-i\theta} \sinh r, \tag{6.2.13}$$

$$S^\dagger(\xi) (a^\dagger)^2 S(\xi) = \left(a^\dagger \cosh r - a e^{-i\theta} \sinh r\right)^2. \tag{6.2.14}$$

Let's first consider the squeezing of a vacuum state $S(\xi)|0\rangle$. The uncertainty of the squeezed state is

$$\sigma(X) = \frac{1}{2}\sqrt{\cosh^2 r + \sinh^2 r - 2\sinh r \cosh r \cos\theta}, \tag{6.2.15}$$

$$\sigma(Y) = \frac{1}{2}\sqrt{\cosh^2 r + \sinh^2 r + 2\sinh r \cosh r \cos\theta}. \tag{6.2.16}$$

When $\theta = 0$,

$$\sigma(X) = \frac{1}{2}e^{-r}, \tag{6.2.17}$$

$$\sigma(Y) = \frac{1}{2}e^{r}. \tag{6.2.18}$$

The state $S(\xi)|0\rangle$ is called the squeezed vacuum state, which the expectation value of the electric field is zero. We can obtain a more general squeeze state by applying both $D(\alpha)$ and $S(\xi)$ on a vacuum state,

$$|\alpha, \xi\rangle \equiv D(\alpha)S(\xi)|0\rangle. \tag{6.2.19}$$

Displacement operators have the relations

$$D^\dagger(\alpha)aD(\alpha) = a + \alpha, \tag{6.2.20}$$

$$D^\dagger(\alpha)a^\dagger D(\alpha) = a^\dagger + \alpha^*, \tag{6.2.21}$$

which add constants and do not change $\sigma(a)$ and $\sigma(a^\dagger)$. This means that $S(\xi)|0\rangle$ and $D(\alpha)S(\xi)|0\rangle$ have the same shapes in the phase space.

6.2.2 Number-State Representations

Let $|\xi\rangle = |0, \xi\rangle$ expressed in the number basis,

$$|\xi\rangle = \sum_n C_n |n\rangle., \tag{6.2.22}$$

where

$$C_n = \begin{cases} 0, & \text{odd}, \\ \frac{i^n}{\sqrt{\cosh r}} \frac{\sqrt{n!}}{2^{n/2}\left(\frac{n}{2}\right)!} e^{in\theta/2} \tanh^{n/2} r, & \text{even}. \end{cases} \tag{6.2.23}$$

For a general squeezed state, $|\alpha, \xi\rangle$, the coefficients are

$$C_n = \exp\left[-\frac{1}{2}|\alpha^2| - \frac{1}{2}(\alpha^*)^2 e^{i\theta}\tanh r\right] \frac{\left(\frac{e^{i\theta}\tanh r}{2}\right)^{n/2}}{\sqrt{n!\cosh r}} H_n\left[\frac{\alpha + \alpha^* e^{i\theta}\tanh r}{\sqrt{2e^{i\theta}\tanh r}}\right]. \tag{6.2.24}$$

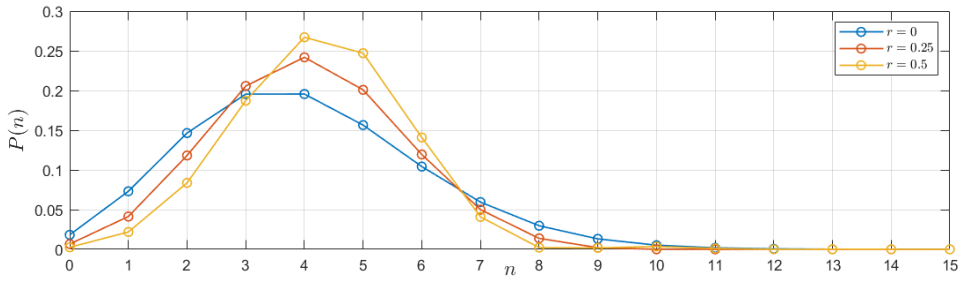

Figure 6.4: Photon counting of squeezed states. $\alpha = 2$

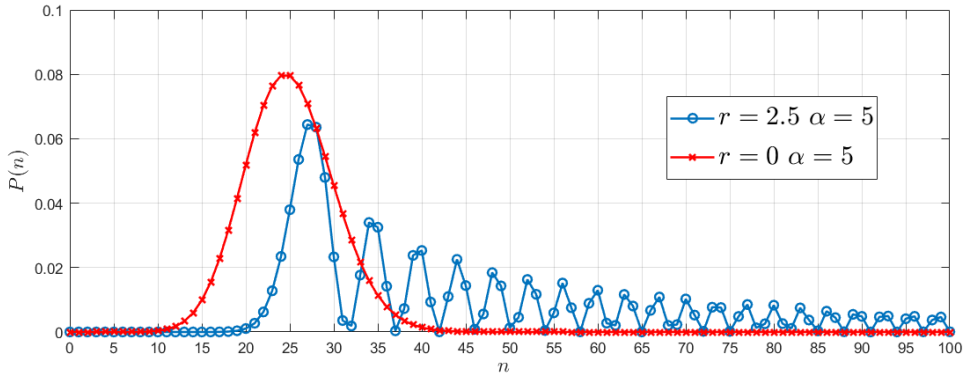

Figure 6.5: Photon counting of squeezed states. $\alpha = 5$