Beam Splitter and Nonclassical Light

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2021

1 Beam Splitters

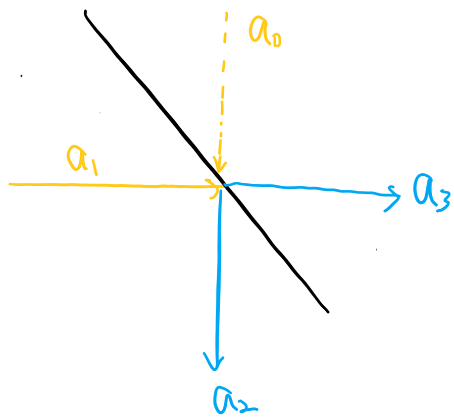

Figure 1: Beam spliter. Quantum descriptions.

Let the right-going photons have the annihilation operator a_1, and the left-going photons have the annihilation operator a_0. A beam splitter is described by the scattering matrix

$$\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}, \tag{1}$$

where $|r|^2 + |t|^2 = 1$. One can check that if

$$[a_0, a_0^\dagger] = 1, \tag{2}$$

$$[a_1, a_1^\dagger] = 1, \tag{3}$$

$$[a_0, a_1^\dagger] = 0, \tag{4}$$

we have

$$[a_i, a_j^\dagger] = \delta_{ij}. \tag{5}$$

The matrix is indeed a scattering matrix

$$U = \begin{pmatrix} t & r \\ r & t \end{pmatrix}, \tag{6}$$

which is unitary.

1.1 Single Photon

The incident state is $|0\rangle_0|1\rangle_1$, and

$$|0\rangle_0|1\rangle_1 = a_1^\dagger|0\rangle_0|0\rangle_1. \tag{7}$$

Aftering incident on the beam splitter, the state becomes

$$\begin{aligned} Ua_1^\dagger|0\rangle_0|0\rangle_1 &= (ra_2 + ta_3)|0\rangle_2|0\rangle_3 \qquad (8)\\ &= r|1\rangle_2|0\rangle_3 + t|0\rangle_2|1\rangle_3, \qquad (9)\end{aligned}$$

which is an entangled state.

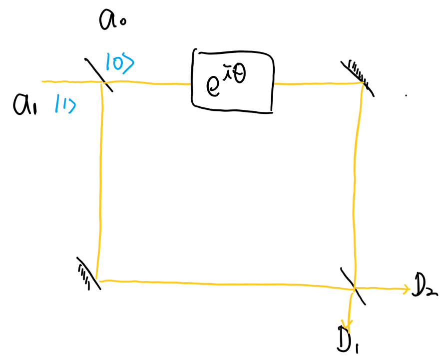

Figure 2: Mach–Zehnder interferometer.

Consider a Mach–Zehnder interferometer with two 50:50 beam splitters of $r = \frac{i}{\sqrt{2}}$ and $t = \frac{1}{\sqrt{2}}$. Let the initial state be $|0\rangle_0|1\rangle_1$. After the first beam splitter, the state becomes

$$\frac{i}{\sqrt{2}}|1\rangle_2|0\rangle_3 + \frac{1}{\sqrt{2}}|0\rangle_2|1\rangle_3. \tag{10}$$

When the state arrives at the second beam splitter, the state becomes

$$\frac{i}{\sqrt{2}}|1\rangle_2|0\rangle_3 + \frac{e^{i\theta}}{\sqrt{2}}|0\rangle_2|1\rangle_3, \tag{11}$$

where θ is a phase shift due to the difference of the two paths. The final state after the second beam splitter is (see the figure: a_2 is the new a_1 and a_3 is the new a_0)

$$\begin{aligned} \begin{pmatrix} a_2 \\ a_3 \end{pmatrix} &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{e^{i\theta}}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} \qquad (12)\\ &= \begin{pmatrix} \frac{(e^{i\theta}-1)}{2} \\ \frac{i(e^{i\theta}+1)}{2} \end{pmatrix} \qquad (13)\end{aligned}$$

The probability at D_1 is $\left|\frac{(e^{i\theta}-1)}{2}\right|^2 = \sin^2\frac{\theta}{2}$. The probability at D_2 is $\left|\frac{i(e^{i\theta}+1)}{2}\right|^2 = \cos^2\frac{\theta}{2}$. This is a more rigorous description of a single-photon interference.

1.2 N-Photons

Let the initial state be $|0\rangle_0|N\rangle_1 = \frac{(a_1^\dagger)^N}{\sqrt{N!}}|0\rangle$. After a beam splitter, the state becomes

$$\frac{(Ua_1^\dagger)^N}{\sqrt{N!}}|0\rangle = \frac{(ta_2^\dagger + ra_3^\dagger)^N}{\sqrt{N!}}|0\rangle. \tag{14}$$

1.3 Coherent States

Let the initial state be $|0\rangle_0|\alpha\rangle_1 = D_1[\alpha]|0\rangle = e^{\alpha a_1^\dagger - \alpha^* a_1}|0\rangle$. After a beam splitter, the state becomes

$$\begin{aligned} e^{\alpha U a_1^\dagger - \alpha^* U^\dagger a_1}|0\rangle &= e^{\alpha t a_2^\dagger - \alpha^* t^* a_2} e^{\alpha r a_3^\dagger - \alpha^* r^* a_3}|0\rangle & (15) \\ &= D_2[t\alpha]D_3[r\alpha]|0\rangle & (16) \\ &= |t\alpha\rangle_2 |r\alpha\rangle_3. & (17) \end{aligned}$$

The input of a coherent state is split into a product of two coherent states. Unlike the single-photon case, this state is not entangled.

Consider a Mach–Zehnder interferometer with two 50:50 beam splitters of $r = \frac{i}{\sqrt{2}}$ and $t = \frac{1}{\sqrt{2}}$. Let the initial state be $|0\rangle_0|\alpha\rangle_1$. After the first beam splitter, the state becomes

$$\left|\frac{i\alpha}{\sqrt{2}}\right\rangle_2 \left|\frac{\alpha}{\sqrt{2}}\right\rangle_3. \tag{18}$$

When the state arrives at the second beam splitter, the state becomes

$$\left|\frac{i\alpha}{\sqrt{2}}\right\rangle_2 \left|\frac{e^{i\theta}\alpha}{\sqrt{2}}\right\rangle_3. \tag{19}$$

where θ is a phase shift due to the difference of the two paths. The final state after the second beam splitter is (see the figure: a_2 is the new a_1 and a_3 is the new a_0)

$$\left|\frac{\left(e^{i\theta} - 1\right)}{2}\right\rangle_2 \left|\frac{i\left(e^{i\theta} + 1\right)}{2}\right\rangle_3. \tag{20}$$

The intensity at D_1 is $|\alpha|^2 \left|\frac{e^{i\theta}-1}{2}\right|^2 = \sin^2\frac{\theta}{2}|\alpha|^2$. The intensity at D_2 is $|\alpha|^2 \left|\frac{e^{i\theta}+1}{2}\right|^2 = \cos^2\frac{\theta}{2}|\alpha|^2$. The two output beams are both coherent states. Thus, the phase θ can be obtained by

$$\frac{I_2 - I_1}{|\alpha|^2} = \cos\theta. \tag{21}$$

However, the amplitudes have uncertainty $\sigma(n) = \sqrt{\bar{n}}$. Thus, the phase has the uncertainty $\sigma(\theta) \sim \frac{1}{\sqrt{\bar{n}}}$. In experiments, we would like to use a coherent light (laser) with a well defined phase and a strong intensity such that the uncertainty in phase is small. But a strong-intensity light may lead to more noise such as radiation pressures, thermo noise, and so on. To solve this deli-ma, lights with small $\sigma(n)$ is used. These lights are non-classical lights.

2 Quadrature Squeezing

The quadrature operators X and Y, satisfy

$$[X,Y] = \frac{i}{2} \tag{22}$$

$$\Rightarrow \sigma(X)\sigma(Y) \geq \frac{1}{4}. \tag{23}$$

The coherent states satisfy the minimum uncertainty equations,

$$\sigma(X)\sigma(Y) = \frac{1}{4} \tag{24}$$

and

$$\sigma(X) = \sigma(Y) = \frac{1}{2}. \tag{25}$$

which is a circle in the phase space. The conditions of a quadrature squeezing are

$$\sigma(X) < \frac{1}{2} \text{ or } \sigma(Y) < \frac{1}{2} \tag{26}$$

while keeping $\sigma(X)\sigma(Y) = \frac{1}{4}$. Pictorially, a squeezed state is an ellipse in the phase space with a area $\frac{\pi}{16}$. Of course, we can squeeze a state in any direction other than X or Y. We can define the rotated quadrature operator as the following

$$X'(\theta) = \frac{ae^{-i\theta} + a^{\dagger}e^{i\theta}}{2}, \tag{27}$$

$$Y'(\theta) = \frac{ae^{-i\theta} - a^{\dagger}e^{i\theta}}{2i}, \tag{28}$$

which represent a coordinate transform of the quadrature operators. Depending on the squeezed axis, we have the following squeezed states. Question: which one has the minimum uncertainty of the photon number?

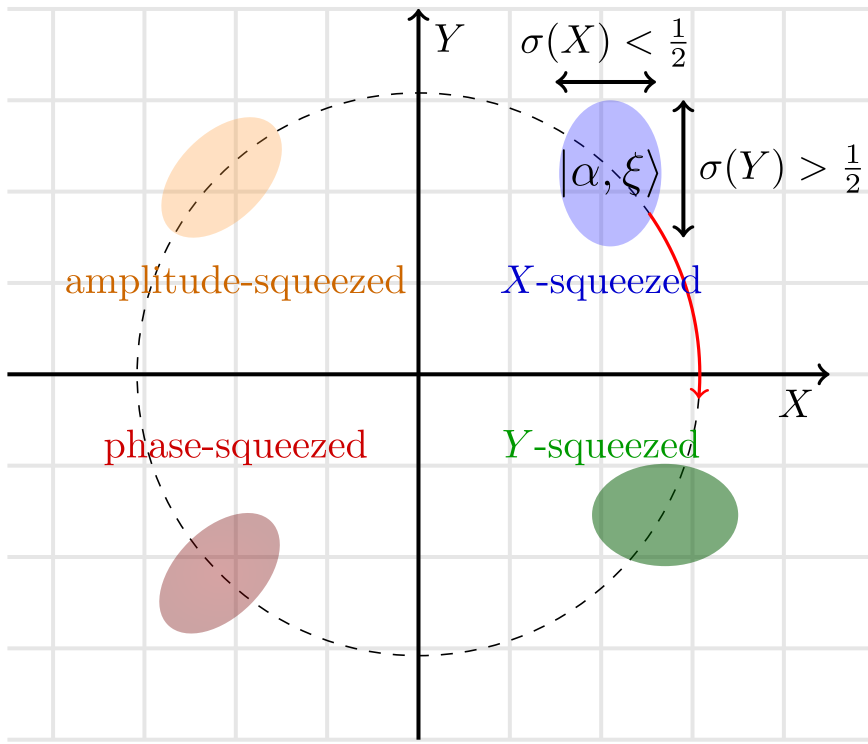

Figure 3: Squeezed States.

2.1 Squeezed Operators

Mathematically, a coherent state is generated by shifting a vacuum state in the phase space. This is done by the displacement operator $D(\alpha)$,

$$|\alpha\rangle = D(\alpha)|0\rangle. \tag{29}$$

We have shown that $D(\alpha)$ is the evolution operator U of a oscillating current source, that is, such a source creates a coherent state.

A squeezed state is generated by a squeeze operator,

$$S(\xi) = \exp\left(\frac{\xi^* a^2 - \xi(a^\dagger)^2}{2}\right), \tag{30}$$

where $\xi = re^{i\theta}$, and r is the squeeze parameter. A squeezed operator is a unitary operator. In principle, a unitary operator correspond a physical process. Observing the quadratic terms of the creation and annihilation operators, it is straightforward to speculate that **the physical processes are nonlinear**. This is because the quadratic terms come from the square of the electric field operators, $\mathbf{E}^2 = \left(\frac{\mathcal{E}a+\mathcal{E}^* a^\dagger}{2}\right)^2$. Squeeze operators have the relations

$$S^\dagger(\xi)S(\xi) = \mathbb{1}, \tag{31}$$
$$S^\dagger(\xi)aS(\xi) = a\cosh r - a^\dagger e^{i\theta}\sinh r, \tag{32}$$
$$S^\dagger(\xi)a^2S(\xi) = \left(a\cosh r - a^\dagger e^{i\theta}\sinh r\right)^2, \tag{33}$$
$$S^\dagger(\xi)a^\dagger S(\xi) = a^\dagger\cosh r - ae^{-i\theta}\sinh r, \tag{34}$$
$$S^\dagger(\xi)(a^\dagger)^2S(\xi) = \left(a^\dagger\cosh r - ae^{-i\theta}\sinh r\right)^2. \tag{35}$$

Let's first consider the squeezing of a vacuum state $S(\xi)|0\rangle$. The uncertainty of the squeezed state is

$$\sigma(X) = \frac{1}{2}\sqrt{\cosh^2 r + \sinh^2 r - 2\sinh r\cosh r\cos\theta}, \tag{36}$$
$$\sigma(Y) = \frac{1}{2}\sqrt{\cosh^2 r + \sinh^2 r + 2\sinh r\cosh r\cos\theta}. \tag{37}$$

When $\theta = 0$,

$$\sigma(X) = \frac{1}{2}e^{-r}, \tag{38}$$
$$\sigma(Y) = \frac{1}{2}e^{r}. \tag{39}$$

The state $S(\xi)|0\rangle$ is called the squeezed vacuum state, which the expectation value of the electric field is zero. We can obtain a more general squeeze state by applying both $D(\alpha)$ and $S(\xi)$ on a vacuum state,

$$|\alpha,\xi\rangle \equiv D(\alpha)S(\xi)|0\rangle. \tag{40}$$

Displacement operators have the relations

$$D^\dagger(\alpha)aD(\alpha) = a + \alpha, \tag{41}$$
$$D^\dagger(\alpha)a^\dagger D(\alpha) = a^\dagger + \alpha^*, \tag{42}$$

which add constants and do not change $\sigma(a)$ and $\sigma(a^\dagger)$. This means that $S(\xi)|0\rangle$ and $D(\alpha)S(\xi)|0\rangle$ have the same shapes in the phase space.

2.2 Number-State Representations

Let $|\xi\rangle = |0,\xi\rangle$ expressed in the number basis,

$$|\xi\rangle = \sum_n C_n|n\rangle., \tag{43}$$

where

$$C_n = \begin{cases} 0, & \text{odd}, \\ \frac{i^n}{\sqrt{\cosh r}} \frac{\sqrt{n!}}{2^{n/2}\left(\frac{n}{2}\right)!} e^{in\theta/2} \tanh^{n/2} r, & \text{even}. \end{cases} \quad (44)$$

For a general squeezed state, $|\alpha, \xi\rangle$, the coefficients are

$$C_n = \exp\left[-\frac{1}{2}|\alpha^2| - \frac{1}{2}(\alpha^*)^2 e^{i\theta} \tanh r\right] \frac{\left(\frac{e^{i\theta}\tanh r}{2}\right)^{n/2}}{\sqrt{n! \cosh r}} H_n\left[\frac{\alpha + \alpha^* e^{i\theta} \tanh r}{\sqrt{2e^{i\theta} \tanh r}}\right]. \quad (45)$$

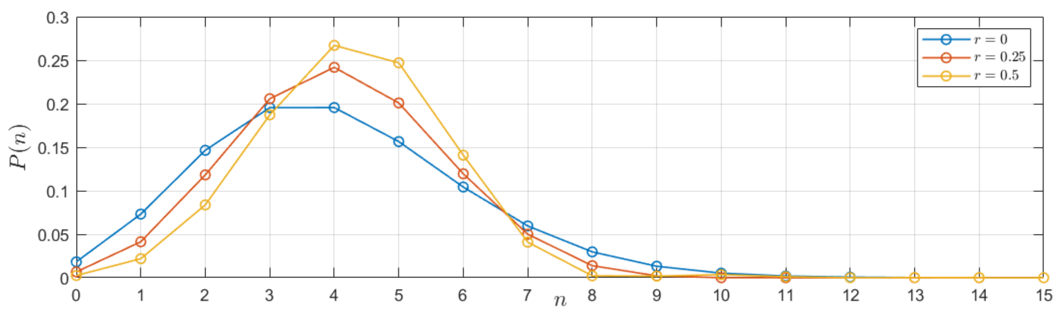

Figure 4: Photon counting of squeezed states. $\alpha = 2$

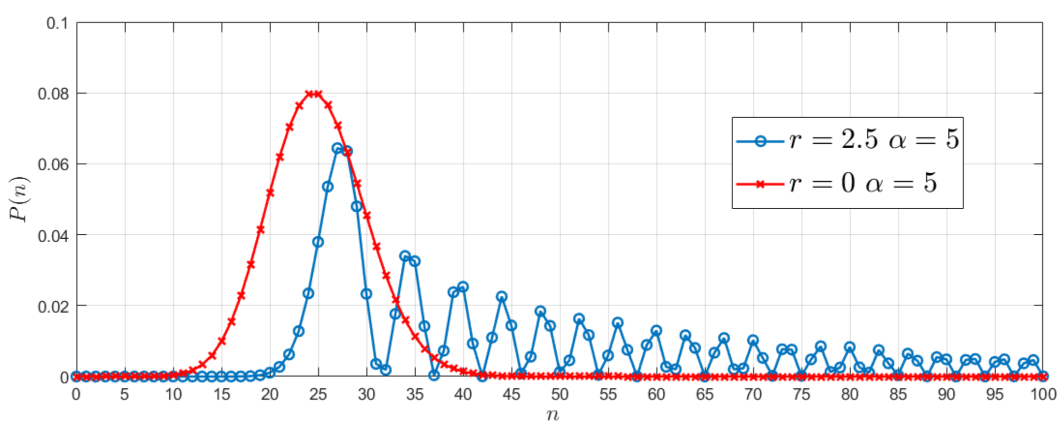

Figure 5: Photon counting of squeezed states. $\alpha = 5$