Quantization of Fields

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The strategy to quantize fields is essentially the same as that for a harmonic oscillator. We think electromagnetic modes as some sorts of oscillations. Every mode with a specific frequency ω behaves as a harmonic oscillator. The quantization of a harmonic oscillator is to make $[x, p] = i\hbar$. Here, x and p are canonical variables of the system. The canonical momentum p is a time-derivative of x. In terms of the creation and annihilation operators, we have the relations

$$x \sim a + a^\dagger, \tag{1}$$
$$p \sim -a + a^\dagger. \tag{2}$$

The Maxwell's equations read

$$\nabla \cdot (\epsilon(\mathbf{r})\mathbf{E}) = 0 \tag{3}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{4}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5}$$
$$\nabla \times \mathbf{B} = \mu(\mathbf{r})\epsilon(\mathbf{r})\frac{\partial \mathbf{E}}{\partial t} \tag{6}$$

1.1 Single Mode

For an electromagnetic mode of a frequency ω, we look for real solutions of the forms,

$$\mathbf{E}_\omega(\mathbf{r}, t) = \boldsymbol{\mathcal{E}}_\omega(\mathbf{r})e^{-i\omega t} + \boldsymbol{\mathcal{E}}^*_\omega(\mathbf{r})e^{i\omega t} \tag{7}$$
$$\mathbf{B}_\omega(\mathbf{r}, t) = \boldsymbol{\mathcal{B}}_\omega(\mathbf{r})e^{-i\omega t} + \boldsymbol{\mathcal{B}}^*_\omega(\mathbf{r})e^{i\omega t}, \tag{8}$$

which satisfy the Maxwell equations. The solutions to the $\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})$ and $\boldsymbol{\mathcal{B}}_\omega(\mathbf{r})$ will depend on the spatial arrangement of the $\epsilon(\mathbf{r})$ and $\mu(\mathbf{r})$. The filed $\mathbf{E}_{\omega 0}(\mathbf{r})$ satisfies

$$\nabla \cdot (\epsilon(\mathbf{r})\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})) = 0, \tag{9}$$
$$\nabla \times (\nabla \times \boldsymbol{\mathcal{E}}_\omega(\mathbf{r})) = \mu(\mathbf{r})\epsilon(\mathbf{r})\omega^2\boldsymbol{\mathcal{E}}_\omega(\mathbf{r}). \tag{10}$$

One can solve the above equations analytically for simple geometries or numerically when geometries are more complicated. Once the $\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})$ is obtained, the magnetic field $\boldsymbol{\mathcal{B}}_\omega(\mathbf{r})$ is given by

$$\nabla \times \boldsymbol{\mathcal{E}}_\omega(\mathbf{r}) = i\omega\boldsymbol{\mathcal{B}}_\omega(\mathbf{r})$$
$$\Rightarrow \boldsymbol{\mathcal{B}}_\omega(\mathbf{r}) = \frac{\nabla \times \boldsymbol{\mathcal{E}}_\omega(\mathbf{r})}{i\omega}. \tag{11}$$

The total energy of the mode is

$$\mathcal{H}_\omega = \int dv \left(\frac{\epsilon(\mathbf{r}) E_\omega^2(\mathbf{r})}{2} + \frac{B_\omega^2(\mathbf{r})}{2\mu(\mathbf{r})} \right), \tag{12}$$

which is similar to

$$\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}. \tag{13}$$

with the analogies

$$x \sim \mathbf{E}_\omega(\mathbf{r}), \tag{14}$$

$$p \sim \mathbf{B}_\omega(\mathbf{r}). \tag{15}$$

It is naturally to speculate[1] that

$$\mathbf{E}_\omega(\mathbf{r}) \sim \boldsymbol{\mathcal{E}}_\omega(\mathbf{r}) a + \boldsymbol{\mathcal{E}}^*_\omega(\mathbf{r}) a^\dagger, \tag{16}$$

$$\mathbf{B}_\omega(\mathbf{r}) \sim -\boldsymbol{\mathcal{B}}_\omega(\mathbf{r}) a + \boldsymbol{\mathcal{B}}^*_\omega(\mathbf{r}) a^\dagger. \tag{17}$$

We define the following field operators

$$\mathbf{E}_\omega(\mathbf{r}) = \frac{\left[\boldsymbol{\mathcal{E}}_\omega(\mathbf{r}) a + \boldsymbol{\mathcal{E}}^*_\omega(\mathbf{r}) a^\dagger\right]}{2}, \tag{18}$$

$$\mathbf{B}_\omega(\mathbf{r}) = \frac{i\left[-\boldsymbol{\mathcal{B}}_\omega(\mathbf{r}) a + \boldsymbol{\mathcal{B}}^*_\omega(\mathbf{r}) a^\dagger\right]}{2} \tag{19}$$

with the normalization conditions

$$\int dv \epsilon |\mathcal{E}_\omega(\mathbf{r})|^2 = \hbar\omega. \tag{20}$$

Plugging Eqs. (18) and (19) in Eq. (12), we obtain the Hamiltonian of a single electromagnetic mode,

$$\mathcal{H}_\omega = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right). \tag{21}$$

All the observables contains the creation and annihilation operator. We can first solve the dynamics of $a(t)$ and obtain all the dynamics. Using the Heisenberg's picture, the equation reads

$$\frac{\partial a}{\partial t} = \frac{i}{\hbar}[\mathcal{H}, a] \tag{22}$$

$$= -i\omega a, \tag{23}$$

[1] You might have the same questions that I had as a student. What are the origins of using a harmonic model to quantize fields? Why is it valid? Why are **E** and **B** the canonical variables? I should say that at least in my opinion, we can not **derive** physics from the first place. All of these steps are hypotheses which are to be examined by experiments. The validities rely on whether the results can explain the observations. To date, it is still the most consistent theory.

which has the solution

$$a(t) = a(0)e^{-i\omega t}. \tag{24}$$

The operator $a^\dagger(t)$ is the hermitian conjugate of $a(t)$,

$$a^\dagger(t) = a^\dagger(0)e^{i\omega t}. \tag{25}$$

Derivation 1: Bonus Credits!

It requires some efforts to derive Eq. (21). We sketch the steps

(a) Plug Eqs. (18) and (19) in Eq. (12).

(b) Show that the integral of the magnetic term is equivalent to the electric term. Replace the magnetic term with Eq. (11). Calculate the integrals with two curls by the integration by parts. Use the identity of vector calculus

$$\int_{\mathcal{V}} dv \mathbf{F} \cdot (\nabla \times \mathbf{A}) = \int_{\mathcal{V}} dv \mathbf{A} \cdot (\nabla \times \mathbf{F}) + \int_{\mathcal{S}} (\mathbf{A} \times \mathbf{F}) \cdot d\mathbf{a}, \tag{26}$$

where $\mathbf{A}$ and $\mathbf{F}$ are arbitrary vector fields. Use Eq. (10) to get rid of the curls.

(c) Use the normalization condition Eq. (20).

Note 1: Quantization fo Fileds

The procedures to quantize a field are:

(a) Find the the eigenmodes (normal modes).

(b) Find the canonical variables.

(c) Define the creation and annihilation operators.

(d) $[a, a^\dagger] = 1$

1.2 Multimode

We have shown how to quantize a single mode of light. We can extend the formulation to multimodes. Let m denote the quantum number of a mode. The total Hamiltonian is

$$\mathcal{H} = \sum_m \hbar\omega_m \left(a_m^\dagger a_m + \frac{1}{2} \right). \tag{27}$$

For example, m can denote the discrete quantum number of a waveguide, or the continuous quantum number $\mathbf{k}$ of a plane wave. If m are discrete numbers, we have the relations

$$[a_m, a_{m'}^\dagger] = \delta_{mm'}. \tag{28}$$

The total field is

$$\mathbf{E}(\mathbf{r}) = \sum_m \mathbf{E}_m(\mathbf{r}). \tag{29}$$

The field operators of the mode m are

$$\mathbf{E}_m(\mathbf{r}) = \frac{\left[\boldsymbol{\mathcal{E}}_m(\mathbf{r})a + \boldsymbol{\mathcal{E}}_m^*(\mathbf{r})a^\dagger\right]}{2}, \tag{30}$$

$$\mathbf{B}_m(\mathbf{r}) = \frac{i\left[-\boldsymbol{\mathcal{B}}_m(\mathbf{r})a + \boldsymbol{\mathcal{B}}_m^*(\mathbf{r})a^\dagger\right]}{2} \tag{31}$$

with the normalization conditions

$$\int dv\epsilon|\mathcal{E}_m(\mathbf{r})|^2 = \hbar\omega_m. \tag{32}$$

The magnetic field operator is given by

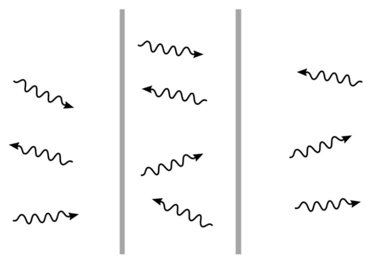
$$\boldsymbol{\mathcal{B}}_m(\mathbf{r}) = \frac{\nabla \times \boldsymbol{\mathcal{E}}_m(\mathbf{r})}{i\omega_m}. \tag{33}$$

Example 1: Casimir Force in a Nutshell!

The vacuum energy of the total Hamiltonian is

$$\left\langle 0 \left| \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2}\right) \right| 0 \right\rangle = \sum_{\mathbf{k}} \frac{\hbar\omega_{\mathbf{k}}}{2}. \tag{34}$$

The integral depends on how many modes there are. The most famous example is the Casimir effect. Consider two parallel metal plates.



The modes in the middle have the wave vector

$$\mathbf{k} = \left(\frac{N\pi}{d}, k_y, k_z\right). \tag{35}$$

Therefore, the vacuum energy of the middle space is

$$E_0(d) = \frac{\hbar}{2} \times 2 \times \left(\int \frac{L_y dk_y}{2\pi} \int \frac{L_z dk_z}{2\pi}\right) \sum_N c\sqrt{k_y^2 + k_z^2 + \frac{N^2\pi^2}{d^2}}. \tag{36}$$

This integral is divergent for any separation d. The potential energy of the system $U(d)$ is defined by

$$U(d) = E_0(\infty) - E_0(d). \tag{37}$$

Although both the two terms are divergent, their difference can be evaluated (See Ref. [1] or Sec. 2.6 of Ref. [2]) as

$$U(d) = \frac{-\pi^2 \hbar c L_y L_z}{720 d^3}. \tag{38}$$

The force per unit area is then

$$\frac{F_c}{L_y L_z} = \frac{1}{L_y L_z}\frac{-\partial U}{\partial d} = -\frac{240\pi^2\hbar c}{d^4}. \tag{39}$$

1.3 Number States

The eigenstates of the photon Hamiltonian, Eq. (27) are the dirext product of the number states $|n_1\rangle \otimes |n_2\rangle$.... which is denoted as $|n_1 n_2...\rangle$. The total energy of the number states $|n_1 n_2...\rangle$ is

$$\langle ...n_2 n_1|\mathcal{H}|n_1 n_2...\rangle = \sum_m \left\langle ...n_2 n_1 \left| \hbar\omega_m \left(a_m^\dagger a_m + \frac{1}{2} \right) \right| n_1 n_2... \right\rangle \tag{40}$$

$$= \sum_m \left(n_m + \frac{1}{2} \right) \hbar\omega_m. \tag{41}$$

For simplicity, we consider a single-mode system in the following. Since the number states are the eigenstates. The expectation values of the observables are static. The expectation values of $\mathbf{E}(t)$ is

$$\langle \mathbf{E}(t) \rangle = \left\langle n \left| \frac{\left[\mathcal{E}_\omega(\mathbf{r}) a + \mathcal{E}_\omega^*(\mathbf{r}) a^\dagger \right]}{2} \right| n \right\rangle = 0. \tag{42}$$

The standard deviation of $\mathbf{E}(t)$ of a number state $|n\rangle$ does not vanish

$$\sigma(\mathbf{E}(t)) = \sqrt{\langle \mathbf{E}(t)^2 \rangle - \langle \mathbf{E}(t) \rangle^2} \tag{43}$$

$$= \sqrt{\langle \mathbf{E}(t)^2 \rangle} \tag{44}$$

$$= |\mathcal{E}_\omega(\mathbf{r})| \sqrt{\frac{n + \frac{1}{2}}{2}} \tag{45}$$

Exercise 1: Standard Deviation

Show Eq. (45). Hint: the operator $\mathbf{E}(t)^2$ is

$$\mathbf{E}(t)^2 = \left(\frac{\left[\mathcal{E}_\omega(\mathbf{r}) a + \mathcal{E}_\omega^*(\mathbf{r}) a^\dagger \right]}{2} \right)^2 \tag{46}$$

$$= \frac{|\mathcal{E}_\omega(\mathbf{r})|^2 (a a^\dagger + a^\dagger a) + \left[\mathcal{E}_\omega(\mathbf{r}) \cdot \mathcal{E}_\omega(\mathbf{r}) a^2 + \mathcal{E}_\omega^*(\mathbf{r}) \cdot \mathcal{E}_\omega^*(\mathbf{r}) (a^\dagger)^2 \right]}{4}. \tag{47}$$

The expectation of $\mathbf{E}(t)^2$ of a number state is

$$\left\langle n \left| \mathbf{E}(t)^2 \right| n \right\rangle. \tag{48}$$

1.4 Plane Waves

The eigenmodes in vacuum are the plane waves with the quantum number $\mathbf{k}$ and s (polarizations). The eigenmode $\boldsymbol{\mathcal{E}}_m(\mathbf{r})$ is

$$\boldsymbol{\mathcal{E}}_m(\mathbf{r}) = \boldsymbol{\mathcal{E}}_{\mathbf{k},s}(\mathbf{r}) \tag{49}$$

$$= \frac{1}{\sqrt{V}}\boldsymbol{\mathcal{E}}_{\mathbf{k},s}e^{i\mathbf{k}\cdot\mathbf{r}} \tag{50}$$

$$= \sqrt{\frac{\hbar\omega}{\epsilon_0 V}}\mathbf{e}_{\mathbf{k},s}e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{51}$$

where V is the volume where the waves exist. $\mathbf{e}_{\mathbf{k},s}$ denotes the two possible polarizations. The total Hamiltonian reads

$$\mathcal{H} = \sum_{\mathbf{k},s}\hbar\omega_{\mathbf{k}}\left(a^\dagger_{\mathbf{k},s}a_{\mathbf{k},s} + \frac{1}{2}\right). \tag{52}$$

The electric and magnetic field operators are

$$\mathbf{E}_{\mathbf{k},s}(\mathbf{r}) = \frac{\left[\boldsymbol{\mathcal{E}}_{\mathbf{k},s}a + \boldsymbol{\mathcal{E}}^*_{\mathbf{k},s}(\mathbf{r})a^\dagger\right]}{2}$$

$$= \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\epsilon_0 V}}\frac{\left[\mathbf{e}_{\mathbf{k},s}e^{i\mathbf{k}\cdot\mathbf{r}}a + \mathbf{e}^*_{\mathbf{k},s}e^{-i\mathbf{k}\cdot\mathbf{r}}a^\dagger\right]}{2}, \tag{53}$$

$$\mathbf{B}_{\mathbf{k},s}(\mathbf{r}) = \frac{\hat{k}}{c}\times\mathbf{E}_{\mathbf{k},s}$$

$$= \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\epsilon_0 V}}\frac{\left[\hat{k}\times\mathbf{e}_{\mathbf{k},s}e^{i\mathbf{k}\cdot\mathbf{r}}a + \hat{k}\times\mathbf{e}^*_{\mathbf{k},s}e^{-i\mathbf{k}\cdot\mathbf{r}}a^\dagger\right]}{2c}. \tag{54}$$

1.5 Thermal Ensemble

An ensemble of photons is specified by the density matrices. The most classic example is a system in the thermal equilibrium. The equilibrium is reached when a photonic system is in contact with a heat reservoir (environment). For a given temperature T, according to statistical mechanics, the probability to occupy a state n is proportional to

$$p(n) \sim e^{-\frac{E_n}{k_B T}}, \tag{55}$$

where k_B is the Boltzmann's constant. Considering the normalization, the probability is

$$p(n) = \frac{e^{-\frac{E_n}{k_B T}}}{\sum_m e^{-\frac{E_m}{k_B T}}} \tag{56}$$

$$= \frac{e^{-\frac{E_n}{k_B T}}}{Z}, \tag{57}$$

with the partition function Z

$$Z = \sum_m e^{-\frac{E_m}{k_B T}}. \tag{58}$$

Thus, the density operator of a thermal ensemble is

$$\begin{aligned}
\rho_{\text{th}} &= \sum_n p(n)|n\rangle\langle n| &(59)\\
&= \frac{\sum_n e^{-\frac{E_n}{k_B T}}|n\rangle\langle n|}{Z} &(60)\\
&= \frac{e^{-\frac{\mathcal{H}}{k_B T}}}{\text{Tr}[e^{-\frac{\mathcal{H}}{k_B T}}]} &(61)
\end{aligned}$$

Exercise 2: Partition Function

Show that the partition function Z of a single mode photonic system is

$$Z = \frac{\exp\left(-\frac{\hbar\omega}{2k_B T}\right)}{1 - \exp\left(-\frac{\hbar\omega}{k_B T}\right)}. \tag{62}$$

Use $E_m = \left(m + \frac{1}{2}\right)\hbar\omega$ in Eq. (58)

The average number of the thermal ensemble is

$$\begin{aligned}
\langle\hat{N}\rangle &= \text{Tr}[\rho_{\text{th}}\hat{N}] &(63)\\
&= \sum_m \langle m|\rho_{\text{th}}\hat{N}|m\rangle &(64)\\
&= \sum_m m\langle m|\rho_{\text{th}}|m\rangle &(65)\\
&= \sum_{m,n} \frac{me^{-\frac{\hbar\omega(n+1/2)}{k_B T}}}{Z}\langle m|n\rangle\langle n|m\rangle &(66)\\
&= \sum_m \frac{me^{-\frac{\hbar\omega(m+1/2)}{k_B T}}}{Z} \quad \text{See Derivation 2} &(67)\\
&= \frac{1}{\exp\frac{\hbar\omega}{k_B T} - 1}, &(68)
\end{aligned}$$

which is the Bose-Einstein distribution.

Derivation 2: Trick of Sums of Series

Let

$$\tilde{Z}(x) = \sum_{m=0}^{\infty} e^{-mx} = \frac{1}{1-e^{-x}}. \tag{69}$$

The trick to calculate the following sums

$$\tilde{Z}_l(x) \equiv \sum_{m=0}^{\infty} m^l e^{-mx}, \tag{70}$$

where l is an integer, is from the relation

$$\tilde{Z}_l(x) = (-1)^l \frac{\partial^l \tilde{Z}}{\partial x^l}. \tag{71}$$

Exercise 3: Standard Derivation of $\hat{N}$

Calculate $\sigma\left(\hat{N}\right)$ of an thermal ensemble of temperature T. Use

$$\sigma\left(\hat{N}\right) = \sqrt{\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2}, \tag{72}$$

$$\langle \hat{N} \rangle = \mathrm{Tr}[\rho_{\mathrm{th}} \hat{N}], \tag{73}$$

$$\langle \hat{N}^2 \rangle = \mathrm{Tr}[\rho_{\mathrm{th}} \hat{N}^2]. \tag{74}$$

1.6 Black-Body Radiation

The average energy of one single mode is $\langle \hat{N} \rangle \hbar\omega$. The density of state of a frequency per unit volume $g(\omega)$ is

$$g(\omega) = \frac{\omega^2}{\pi^2 c^3}. \tag{75}$$

The average energy density $U(\omega)$ is

$$U(\omega) = \langle \hat{N} \rangle \hbar\omega g(\omega) \tag{76}$$

$$= \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{\exp \frac{\hbar\omega}{k_B T} - 1}. \tag{77}$$

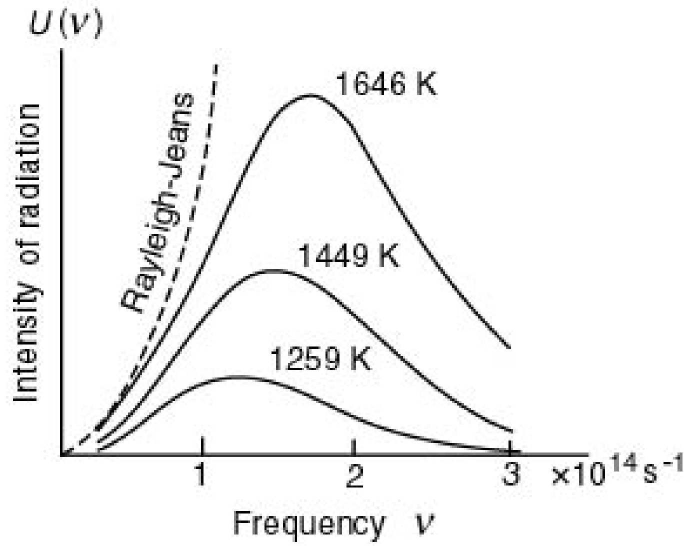


Figure 1: Energy density of a thermal ensemble of photons. Image Source

Its classical analog is the Rayleigh-Jeans formula

$$U(\omega) = g(\omega) k_B T = \frac{\omega^2}{\pi^2 c^3} k_B T, \tag{78}$$

which leads to the ultraviolet catastrophe of the classical physics.

Derivation 3: Density of States

A cuboid has the side lengths L_x, L_y and L_z. The allowed wave vectors are

$$k_x = \frac{2\pi l_x}{L_x} \tag{79}$$

$$k_y = \frac{2\pi l_y}{L_y} \tag{80}$$

$$k_z = \frac{2\pi l_z}{L_z} \tag{81}$$

where l_x, l_y and l_z are integers. The change of the total number m of modes is

$$\Delta m = 2\Delta l_x \Delta l_y \Delta l_z = 2\left(\frac{L_x L_y L_z}{(2\pi)^3}\right)\Delta k_x \Delta k_y \Delta k_z, \tag{82}$$

where the factor 2 accounts for the polarizations. In the continuum limit, it becomes

$$\frac{dm}{V} = \left(\frac{1}{4\pi^3}\right) dk_x dk_y dk_z \tag{83}$$

$$= \frac{1}{4\pi^3} 4\pi k^2 dk \tag{84}$$

$$= \frac{1}{\pi^2}\frac{\omega^2 d\omega}{c^3}, \tag{85}$$

$$\Rightarrow g(\omega) \equiv \frac{1}{V}\frac{dm}{d\omega} = \frac{\omega^2}{\pi^2 c^3}. \tag{86}$$

1.7 Quadrature Operators

We have applied the ideas of a harmonic oscillator to quantize fields. The conjugate variables of a particle are x and p, which are just numbers. Unlike a particle, an photon has field operators $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$, which have values at every position. The similarities of them are the creation and annihilation operators a and $a^\dagger$. It is then useful to define the **dimensionless** operators for photons. We introduce the quadrature operators,

$$X = \frac{a + a^\dagger}{2}, \tag{87}$$

$$Y = \frac{a - a^\dagger}{2i}. \tag{88}$$

The operator X is the dimensionless position operator, and the operator Y is the dimensionless momentum. They have the relation

$$[X, Y] = \frac{i}{2}. \tag{89}$$

Using the generalized uncertainty relation, we obtain

$$\sigma(X)\sigma(Y) \geq \frac{|\langle [X, Y]\rangle|}{2} = \frac{1}{4}. \tag{90}$$

The electric field operator of a mode m is rewritten as

$$\mathbf{E}_m(\mathbf{r}) = \mathrm{Re}[\mathcal{E}_m(\mathbf{r})]X - \mathrm{Im}[\mathcal{E}_m(\mathbf{r})]Y. \tag{91}$$

In the case of plane waves, the electric field operator of a mode $\{\mathbf{k}, s\}$ is

$$\mathbf{E}_{\mathbf{k},s}(\mathbf{r}) = \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\epsilon_0 V}}\Big\{\mathrm{Re}[\mathbf{e}_{\mathbf{k},s}]\cos(\mathbf{k}\cdot\mathbf{r})X - \mathrm{Im}[\mathbf{e}^*_{\mathbf{k},s}]\sin(\mathbf{k}\cdot\mathbf{r})Y\Big\}. \tag{92}$$

References

[1] P. W. Milonni and M.-L. Shih, *Contemporary Physics*, volume 33, number 5, pages 313-322, 1992

[2] C. Gerry and P. Knight, *Introductory Quantum Optics*, Cambridge University Press, 2005